To:	D. W. Mueller, Jr., Ph.D., P.E.
From:	Mason Averill
Subject:	Special Problem 2
Date:	3 September 2020

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<u>Purpose:</u>

The purpose of this memo is to communicate the methodology and results of Special Problem 2 for ME-544: Modeling and Simulation of Mechanical Engineering Systems, completed September 3rd 2020.

<u>Purpose and Scope of Assignment:</u>

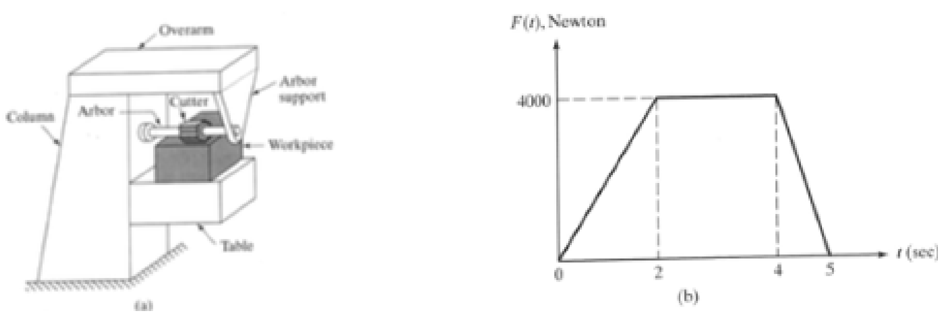
The purpose of this assignment was to determine the equation of motion for a milling machine mounted on an elastic foundation undergoing a harmonic disturbance (represented by Y(t)) with a supplemental external force applied to the milling machine (represented by F(t)). In addition, the position, velocity, and acceleration of the milling machine from 0 to 10 seconds was to be found. The magnitude of the variation of the position, velocity, and acceleration over the same time period was also to be found.

The problem's given information is shown below by Figure 1.

A milling machine is supported on an elastic foundation as shown in the figure below. A external force, $F(t)$, is applied to the machine during the metal cutting operation. In addition, the floor on which the machine is mounted is subjected to a harmonic disturbance, $y(t)$, due to the operation of an unbalanced engine in the vincinity of the milling machine. The equation of vibration is given by

$$m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky + F(t)$$

If $m = 5000$ kg, $c = 1$ kN-s/m, $k = 1$ MN/m, $y(t) = 2\sin(200\pi t)$ mm, and $F(t)$ is given below.



milling machine — external applied force

Figure 1-Problem Information

Mathematical Modeling of Problem:

First, the equation of motion of the milling machine was derived, as shown in Figure 2. Next, a function was developed to model the external force as a function of time, as shown in Figure 3. To accomplish this, the Heaviside mathematical function was utilized. This function acts like a switch, allowing one to generate an equation representative of discontinuous functions such as F(t) shown in Figure 1. The resulting F(t) function modeled using the Heaviside function turns "on" the linearly increasing force from 0 seconds to 2 seconds, then turns it "off". Next it turns "on" the constant force applied from 2 seconds to 4 seconds, then it turns it "off" at 4 seconds. Finally, it turns "on" the linearly decreasing force from 4 seconds to 5 seconds, then turns it "off" at 5 seconds. The Heaviside function allows for excellent flexibility when modeling a discontinuous function such as F(t).

Now that F(t) was modeled using the Heaviside function, the solution of the equation of motion expression could begin. First, the Laplace transform of the Y terms on the right hand side of the equation of motion expression were found symbolically. Next, the Laplace transform of the X terms on the left hand side of the equation of motion expression were found symbolically. Finally, terms were reorganized and the equation of motion was solved for X(s), which is representative of the displacement of the milling machine as a function of frequency. This is all the mathematical modeling that was necessary, as the process of actually taking the Laplace transform of this function as well as taking the inverse Laplace transform was done in MATLAB. Once solved, the resulting expression is representative of the position of the milling machine as a function of time.

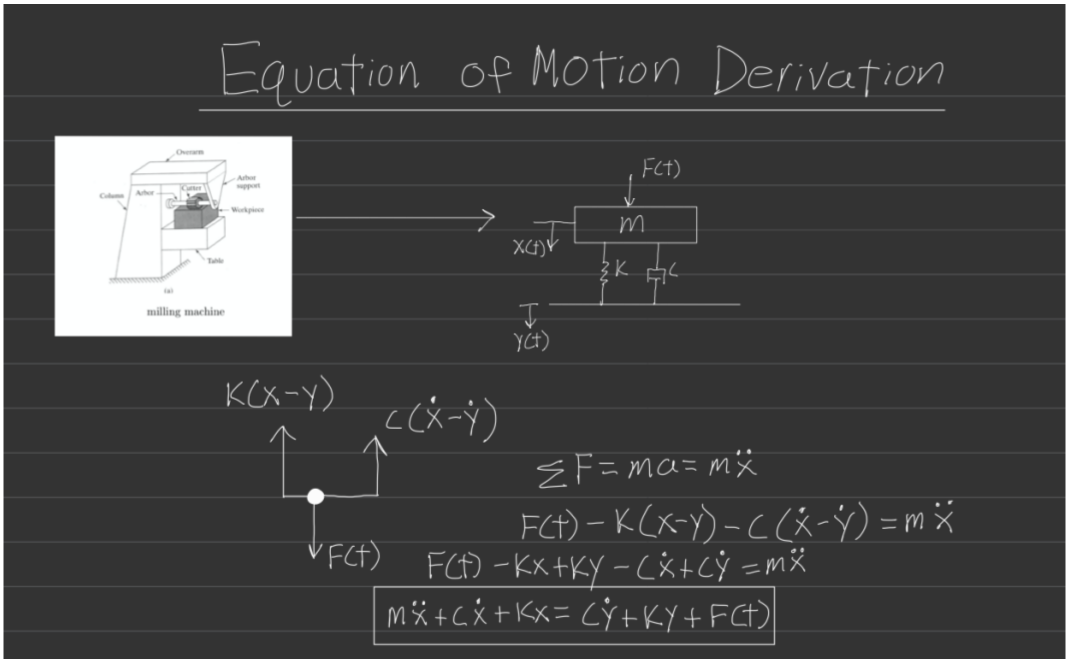


Figure 2-Equation of Motion Derivation

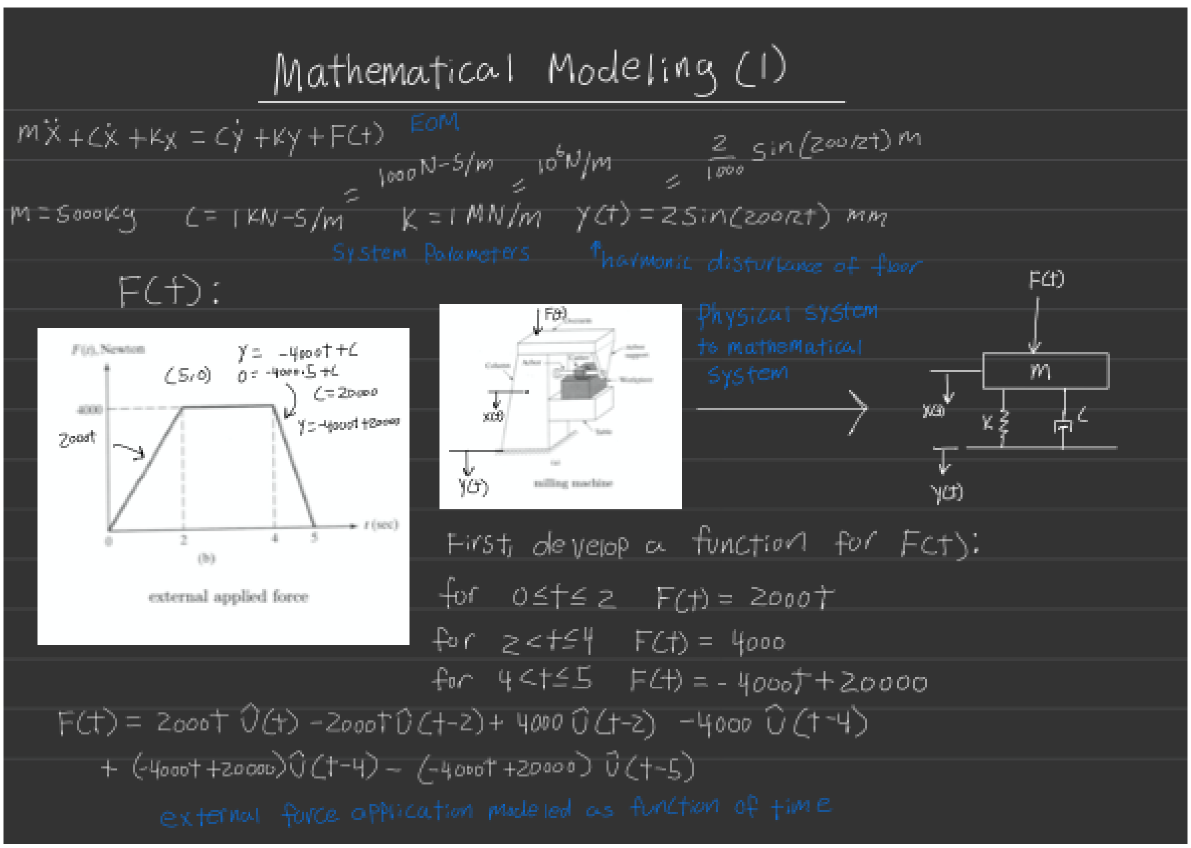


Figure 3-Mathematical Modeling 1

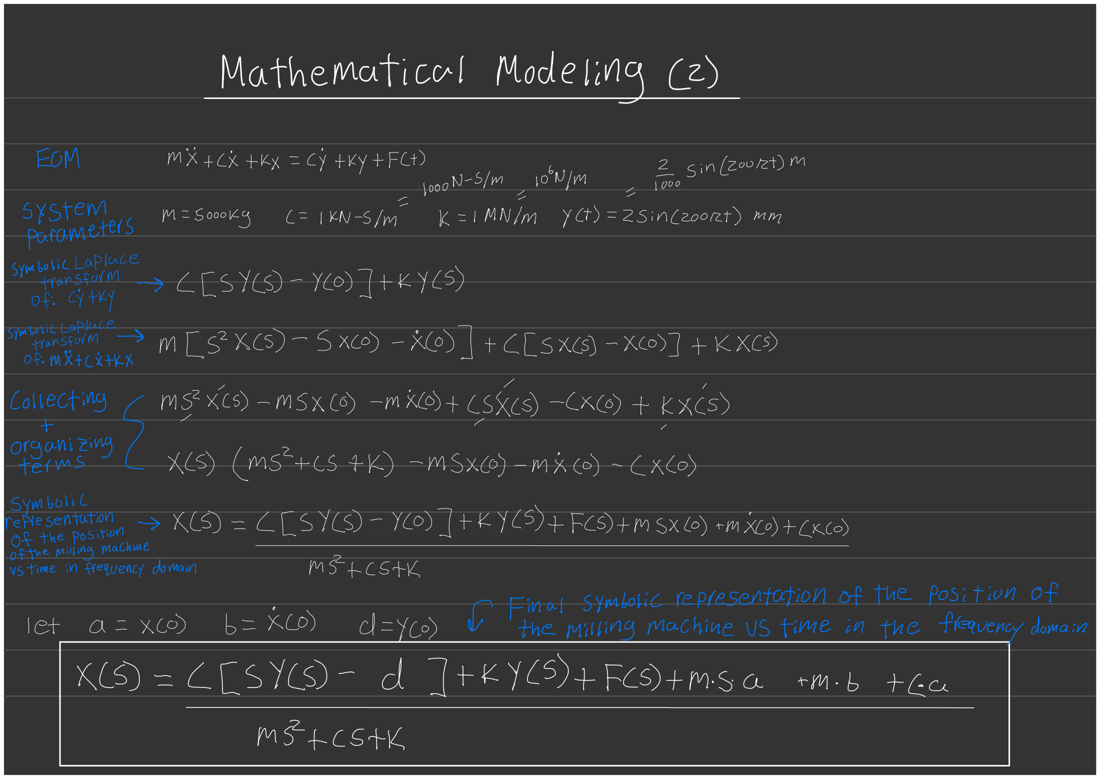


Figure 4-Mathematical Modeling 2

<u>Development of Simulation:</u>

A MATLAB script was written to solve this problem. The MATLAB script accepts input parameters of:

- m: The mass of the milling machine
- c: The damping coefficient of the milling machine
- k: The equivalent spring constant of the milling machine
- y(t): The function modeling the harmonic disturbance of the foundation the milling machine sits on
- x(0): The initial position of the milling machine at a time equal to 0 seconds
- $\dot{x}$(0): The initial velocity of the milling machine at a time equal to 0 seconds
- y(0): The initial position of the foundation at a time equal to 0 seconds
- F(t): The function modeling the input force on the milling machine as a function of time

The MATLAB script allows for easy modification of all of the above parameters, and has an easy way in which to switch the direction of the input force on the machine, in the direction of x or opposed to the x direction.

With input parameters given, the MATLAB script performs the following operations:

1. The Laplace transform of F(t) is found
2. The Laplace transform of the Y terms on the right hand side of the equation of motion expression are found, with the initial condition associated with Y
3. The equation of motion is solved for X(s), with initial conditions for the position and velocity of the milling machine considered
4. An inverse Laplace transform of X(s) is taken, which yields an equation modeling the position of the milling machine as a function of time, with all of the input parameters considered
5. A linearly spaced row vector representative of time in seconds is generated, with user-specified resolution and the maximum time of interest for the equation of motion of the milling machine
6. The symbolic result from the inverse Laplace transform is converted to a numeric row vector, with each element being representative of an increment in time corresponding to the row vector created in 5.
7. The velocity of the milling machine is found by taking the difference between elements contained in the position array and dividing by the difference in time between the elements in the position array
8. The acceleration of the milling machine is found in a similar manner, just using the velocity of the milling machine rather than the position of the milling machine
9. The position, velocity, and acceleration of the milling machine is plotted with a user-specified range of values for the magnitude of position, velocity, and acceleration
10. The variation in the position, velocity, and acceleration is found and printed to the console

Verification/Validation of Model:

First, the model was tested with the given input parameters for the spring constant equivalent (k), damping coefficient (c), mass (m), F(t), and with all initial conditions having a value of 0. The results with these input parameters can be seen in Figures 5-7.

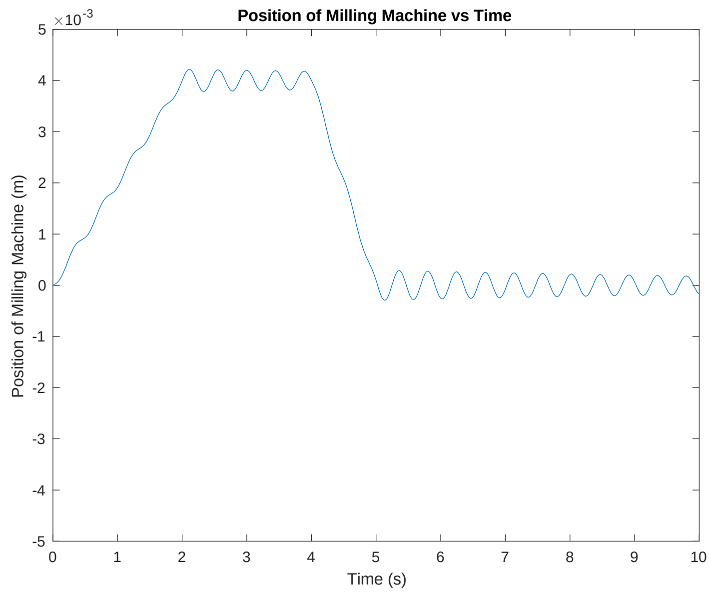


Figure 5-Position of Milling Machine from 0 to 10 Seconds, Given Input Parameters, Initial Conditions All 0

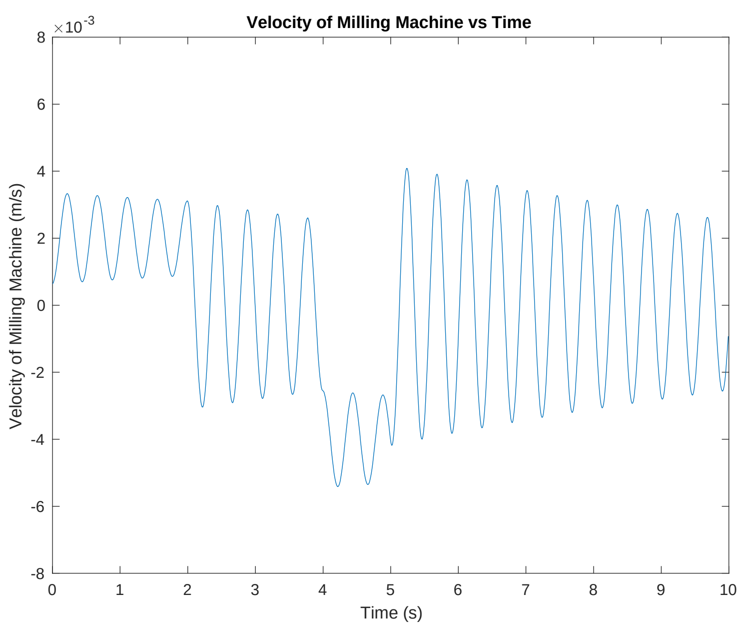


Figure 6-Velocity of Milling Machine from 0 to 10 Seconds, Given Input Parameters, Initial Conditions All 0

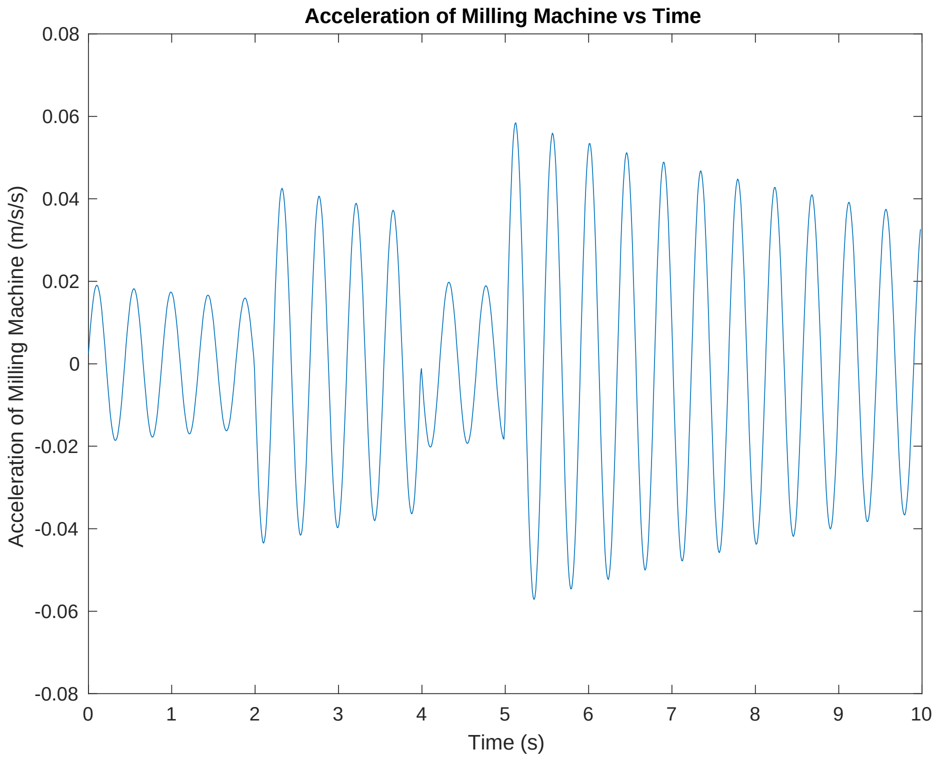


Figure 7-Acceleration of Milling Machine from 0 to 10 Seconds, Given Input Parameters, Initial Conditions All 0

Figures 5-7 demonstrate that not only does the model behave as expected (one would expect the position vs time plot to resemble the input force vs time plot, but with oscillation due to the harmonic disturbance of the foundation), but also by comparing the position, velocity, and acceleration plots it is evident that they are in fact representative of derivatives of one another.

Figure 8 shows a position vs time plot with all of the same input parameters, except the damping coefficient was increased by an order of magnitude. One would expect the position vs time plot to reach steady state more rapidly because the force resisting motion is directly proportional to the damping coefficient, which it does.

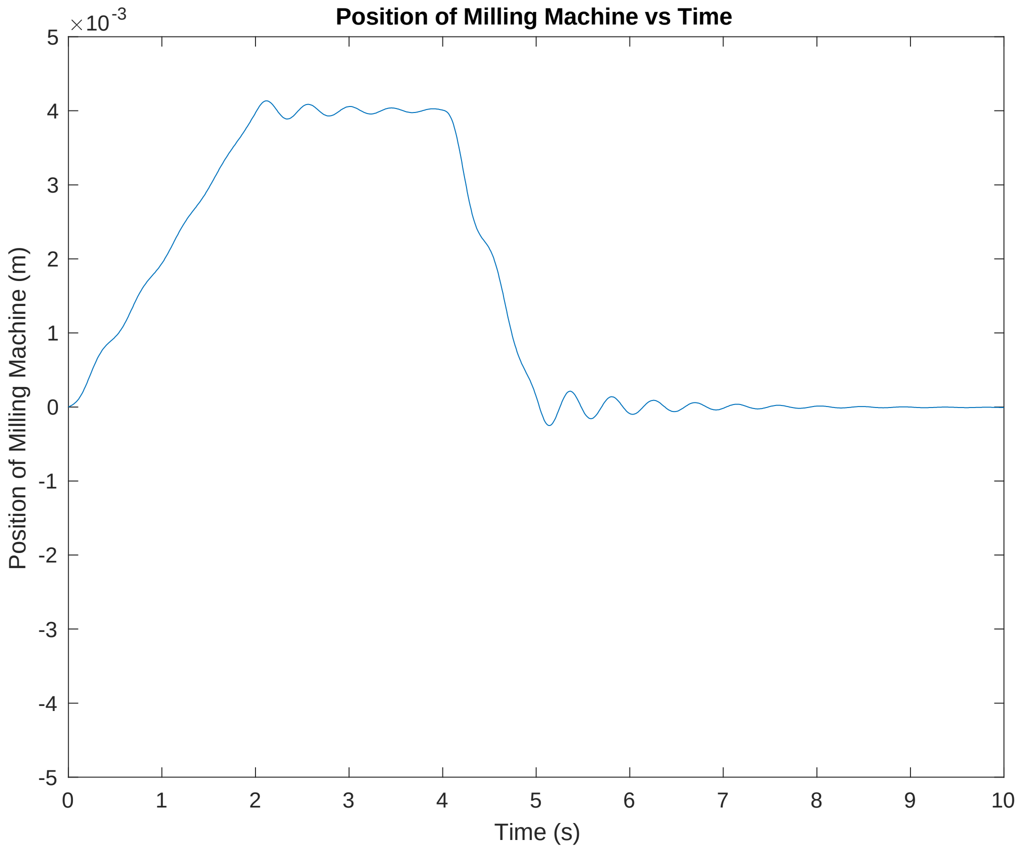


Figure 8-Position of Milling Machine from 0 to 10 Seconds, Damping Coefficient Increase Response Verification

Next, the model's resonant frequency response was tested. When the frequency of the input excitation is equal to the square root of the milling machine's mass divided by its equivalent spring constant, resonant frequency occurs. In this case, the damping coefficient was set to 0. Due to this, one would expect the magnitude of the position and acceleration to tend towards infinity with sufficiently large enough time.

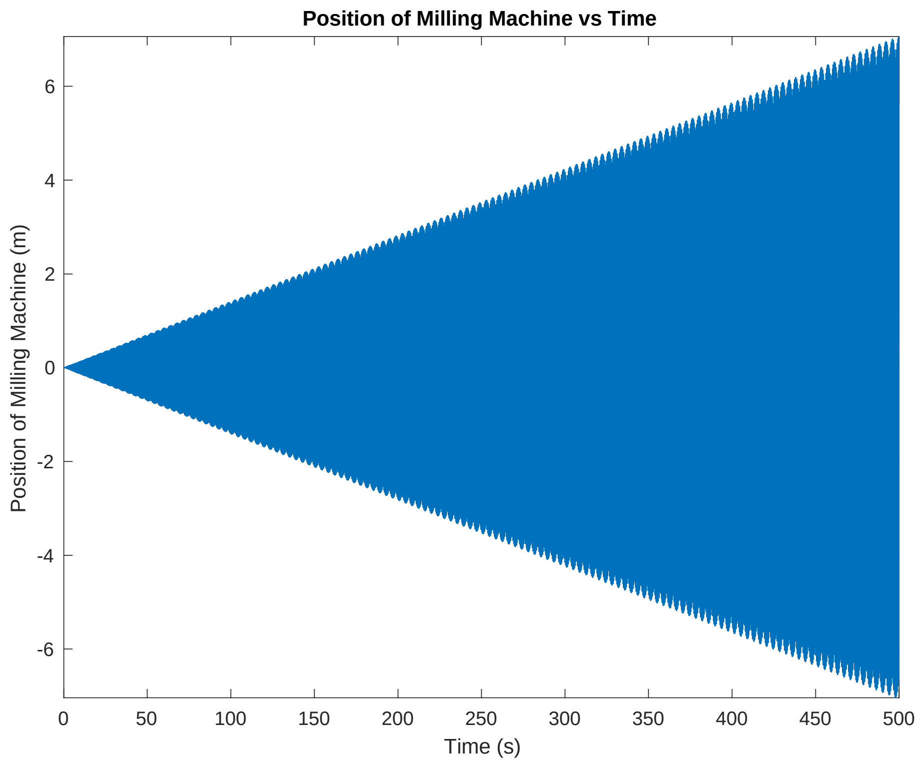


Figure 9-Position of Milling Machine vs Time, Resonant Frequency, Damping Coefficient of 0

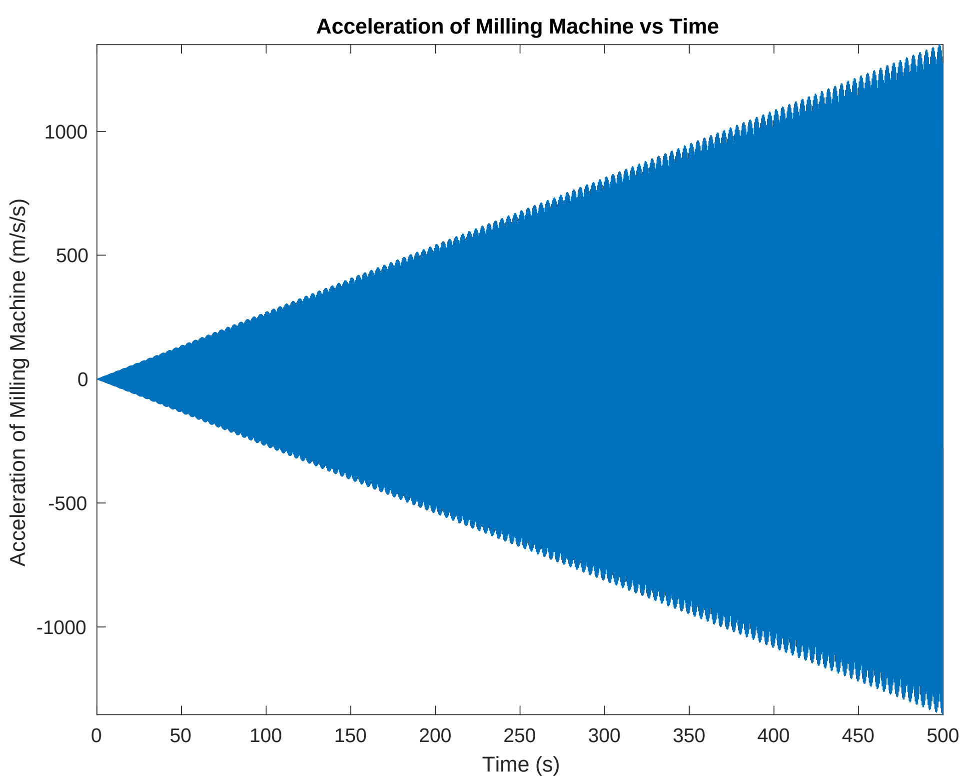


Figure 10-Acceleration of Milling Machine vs Time, Resonant Frequency, Damping Coefficient of 0

As can be seen by Figures 9 and 10, the magnitude of the position and acceleration increase as time increases. At a sufficiently large time both of their amplitudes would tend towards infinity. Note that at a time of just 8 minutes the position of the milling machine is deviating by over 6 meters and the acceleration is over 1000m/s/s. This is why considering resonant frequency is of such grave importance when designing anything subjected to an oscillatory input.

Next, the input excitation frequency was kept the same, still at resonant frequency, but the damping coefficient was returned to the given value of 1000N-s/m. One would expect the magnitude of the position and acceleration plot to eventually remain constant after a sufficiently high enough velocity is achieved (the damper's force is directly proportional to velocity, so with large velocity the force resisting excitation becomes large as well).

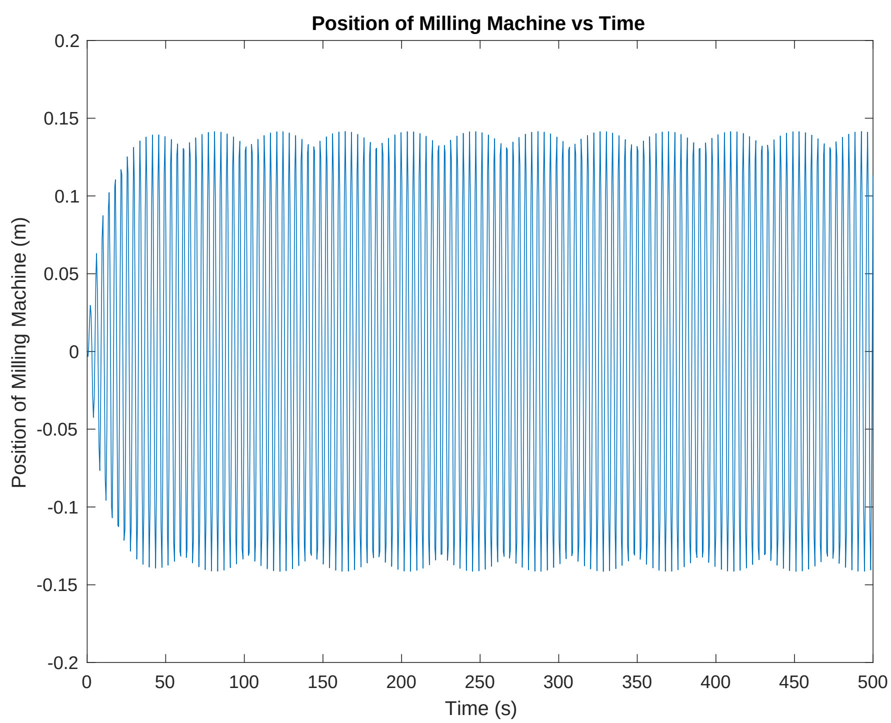


Figure 11-Position of Milling Machine vs Time, Resonant Frequency, Damping Coefficient of 1000 N-s/m

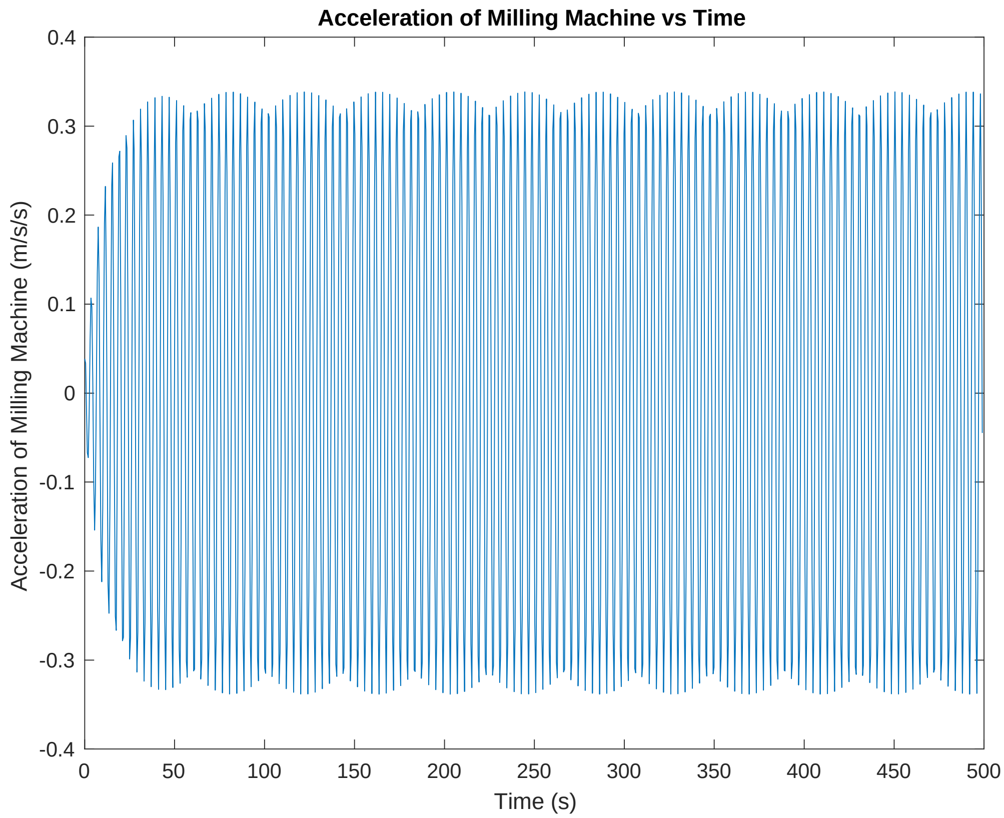


Figure 12-Acceleration of Milling Machine vs Time, Resonant Frequency, Damping Coefficient of 1000 N-s/m

As can be seen by Figures 11 and 12, the model behaves exactly as expected.

Next, the initial conditions were all set to 0.005 (representative of position in meters and velocity in m/s) and the input excitation frequency was changed back to the given original. One would expect the results of this simulation to resemble those attained when the initial conditions were all 0, just slightly shifted up at time equal to 0 and the response would be increasingly less altered with increasing time.

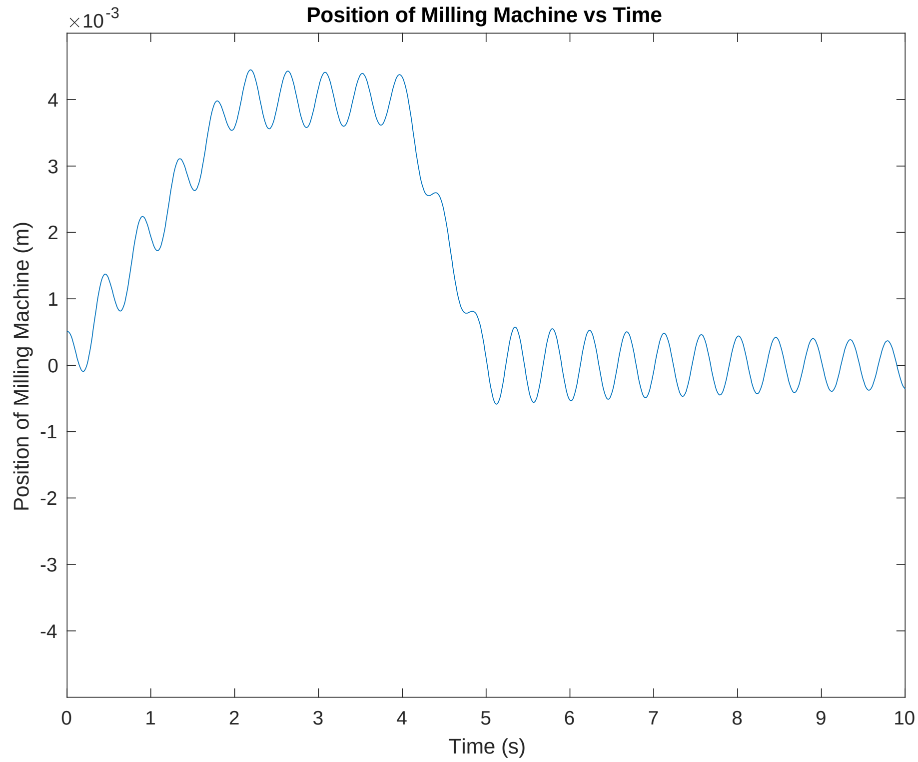


Figure 13-Position of Milling Machine from 0 to 10 Seconds, Given Input Parameters, Initial Conditions All 0.005

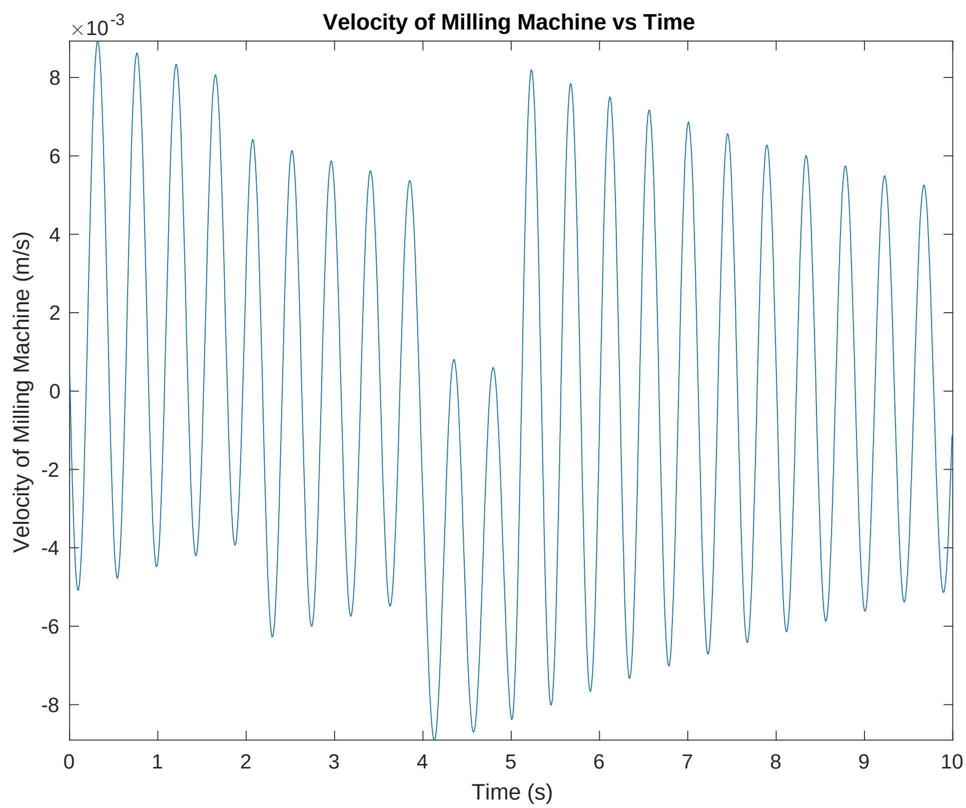


Figure 14-Velocity of Milling Machine from 0 to 10 Seconds, Given Input Parameters, Initial Conditions All 0.005

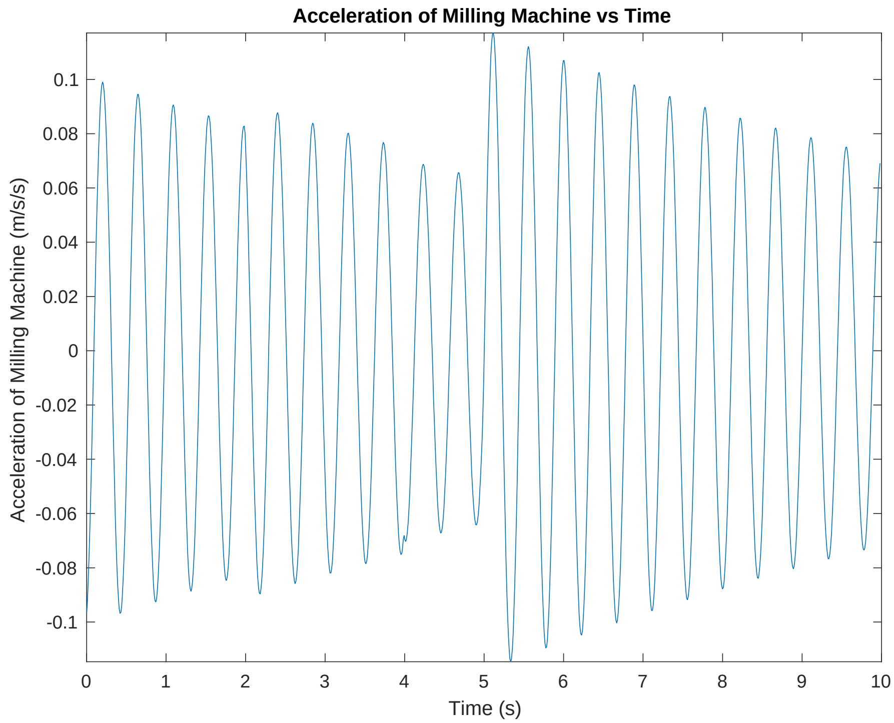


Figure 15-Acceleration of Milling Machine from 0 to 10 Seconds, Given Input Parameters, Initial Conditions All 0.005

Comparing Figures 13-15 with Figures 5-7 demonstrates that, again, the model behaves as expected.

The position of the milling machine as a function of time was also found using the state space function in MATLAB. The results of this solution and the results of the solution using the aforementioned methodology were plotted in the same figure, shown by Figure 16.

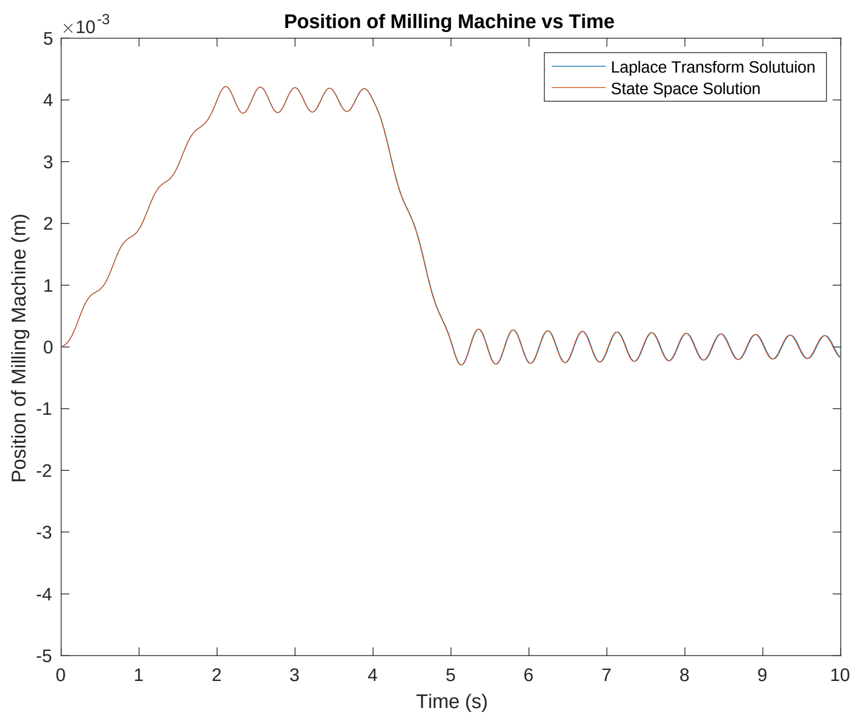


Figure 16-Laplace Transform Solution Method vs State Space Solution Method

By viewing Figure 16 it is evident that both solution methods yield the same result, further verifying the solution.

Execution of Simulation to Determine Position, Velocity, and Acceleration of Milling Machine Over A Time Period From 0 To 10 Seconds As Well As the Magnitude of the Variation in Position, Velocity, and Acceleration Over the Same Time Period:

All prior Figures shown have been at resolutions at which identifying trends and responses is much easier to view. For example, Figures 17-18 depict two plots of the given function describing the disturbance of the foundation with 1,000 increments considered (100 data points considered per second), and 100,000 increments considered (10,000 data points considered per second). The reason for the vast difference in appearance between plots is that the function describing the harmonic disturbance of the foundation has a frequency of 100 cycles per second. Using only 1,000 increments means that data is collected for every 360 degrees of oscillation for the disturbance function. Using 100,000 increments means that data is collected for every 3.6 degrees of oscillation for the disturbance function.

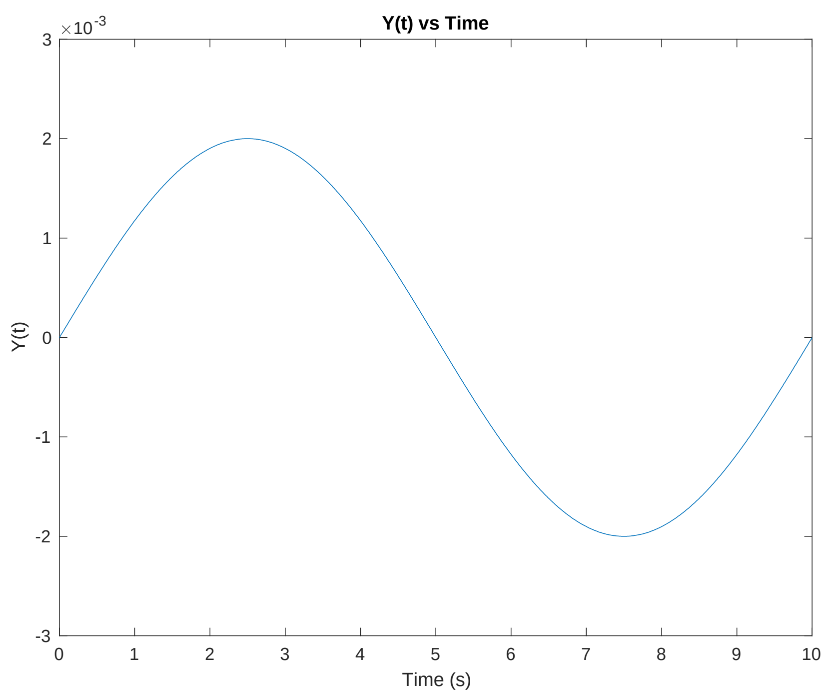


Figure 17-Y(t) from 0 to 10 Seconds, 1,000 Increments Considered

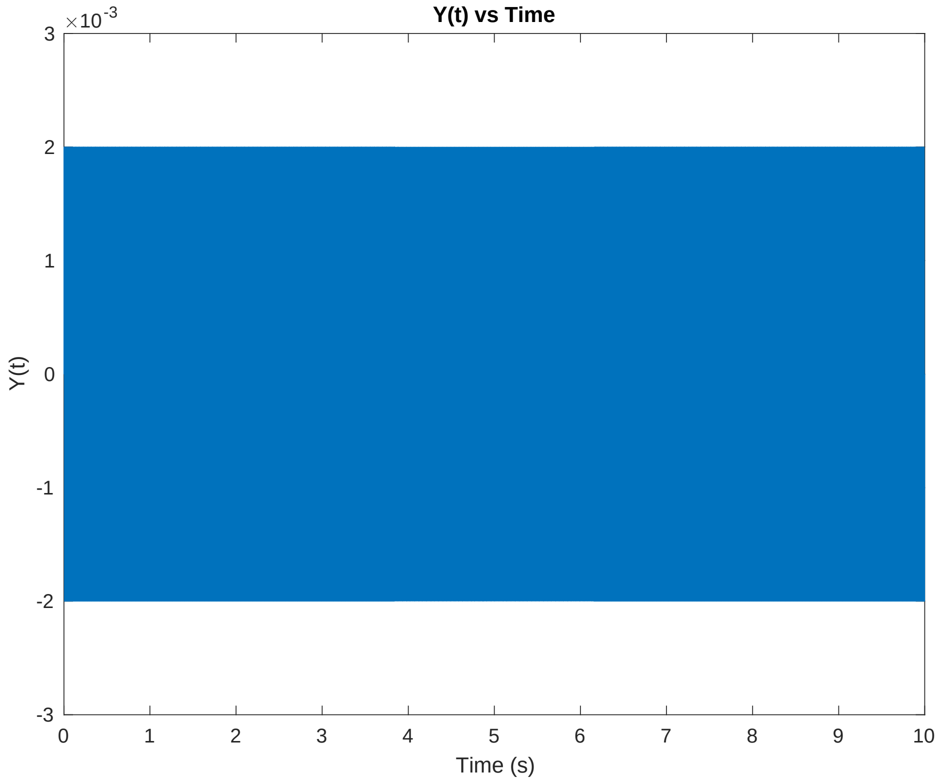


Figure 18-Y(t) from 0 to 10 Seconds, 100,000 Increments Considered

As you can see, it is hard to visualize what is occurring for Y(t) over such a relatively large margin of time due to its high frequency. Figure 18 ends up resembling a rectangle more than anything, but if you zoom in on the Figure it is clear that it is still oscillatory in nature, this is demonstrated by Figure 19.

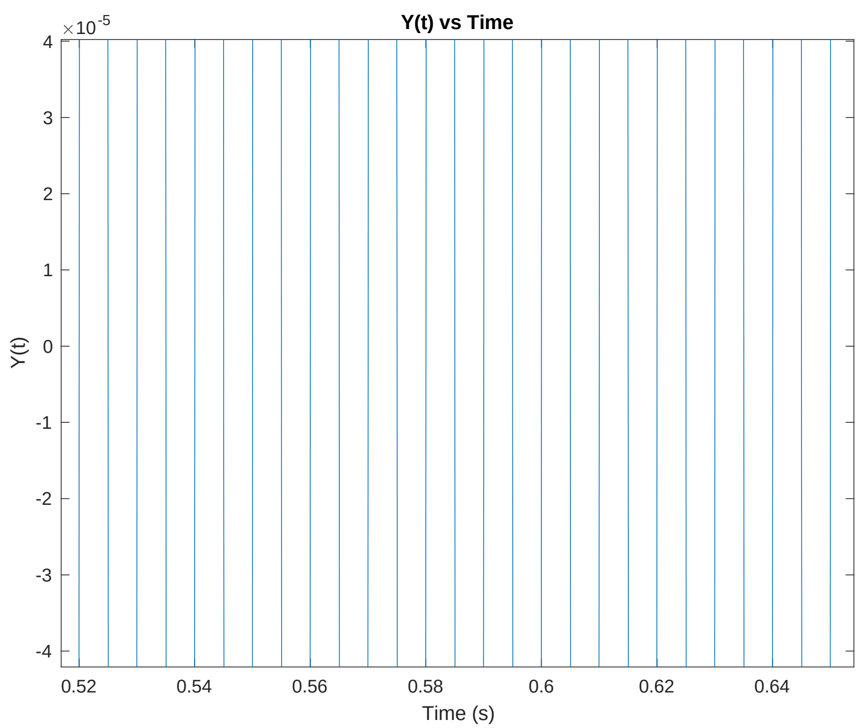


Figure 19-Y(t) Zoomed In

With this considered, it is clear that in order to accurately capture the position, velocity, and acceleration of the milling machine vs time, far more than 1,000 increments need to be utilized. Figures 20-22, respectively, depict the position, velocity, and acceleration of the milling machine vs time. These plots were generated with 100,000 increments considered, capturing every 3.6 degrees of the oscillation of the harmonic disturbance in the foundation.

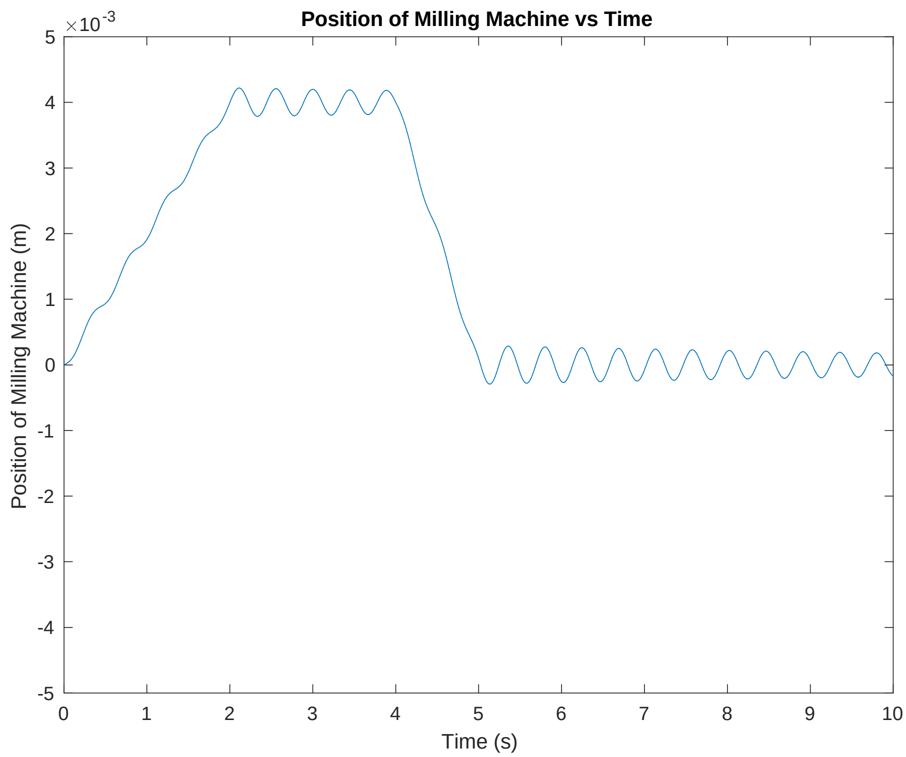


Figure 20-Position of Milling Machine vs Time, Given Input Parameters, All Initial Conditions 0, 100,000 Increments Considered

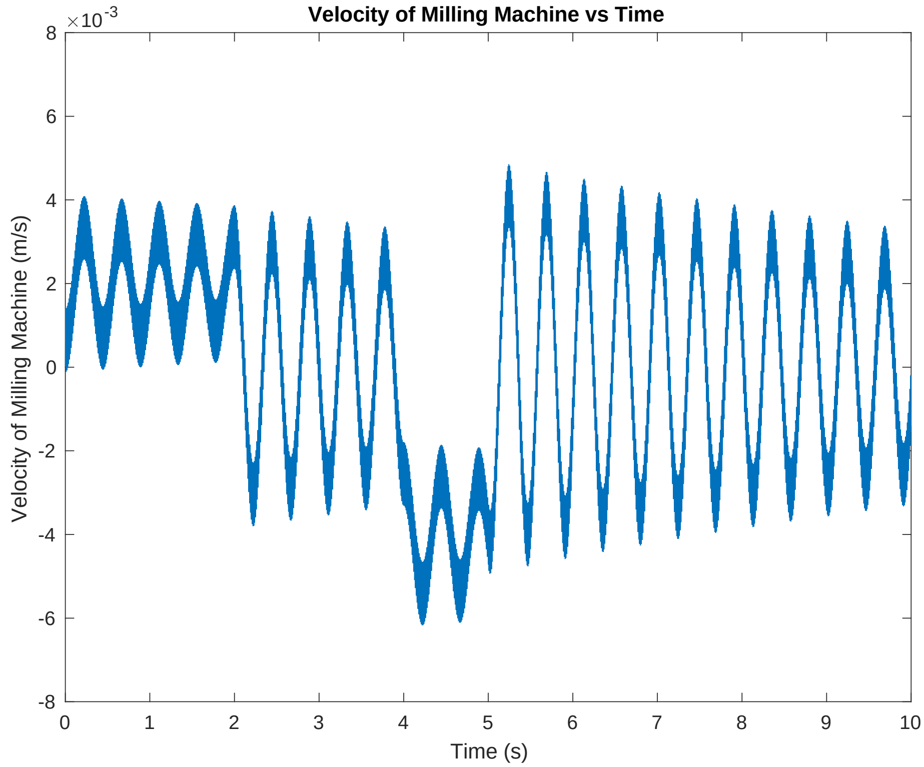


Figure 21-Velocity of Milling Machine vs Time, Given Input Parameters, All Initial Conditions 0, 100,000 Increments Considered

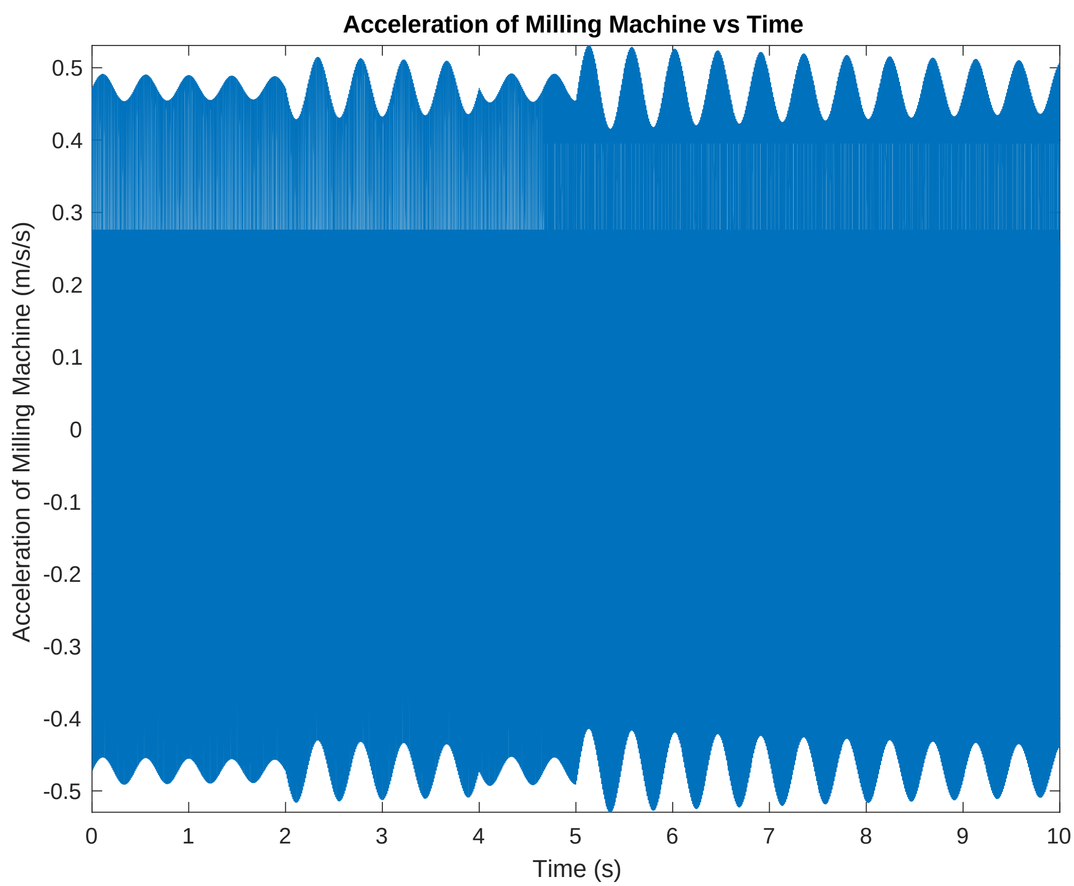


Figure 22-Acceleration of Milling Machine vs Time, Given Input Parameters, All Initial Conditions 0, 100,000 Increments Considered

This resulted in the following values for the magnitude of variation in position, velocity, and acceleration, respectively:

1.) 0.0045125518001151374014590089700505 (m)
2.) 0.011015160371744891348866346447721 (m/s)
3.) 1.0604410940506752147882707504323 (m/s/s)

Next, 500,000 increments were considered. This means the disturbance of the foundation was captured at every 0.72 degrees of its oscillation. Figures 23-25 depict these results.

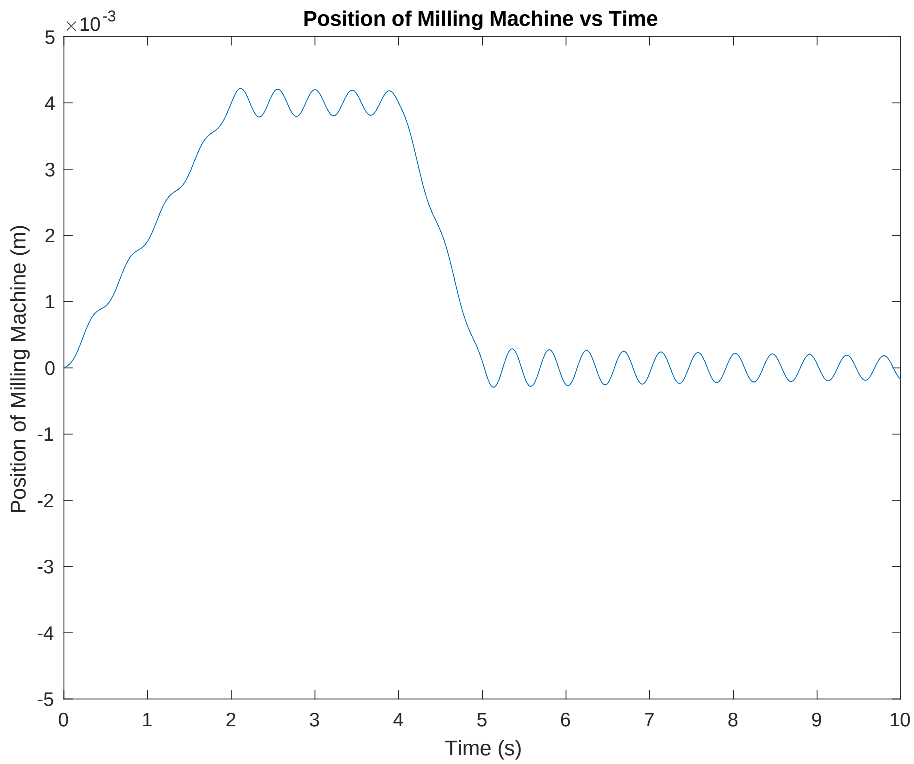


Figure 23-Position of Milling Machine vs Time, Given Input Parameters, All Initial Conditions 0, 500,000 Increments Considered

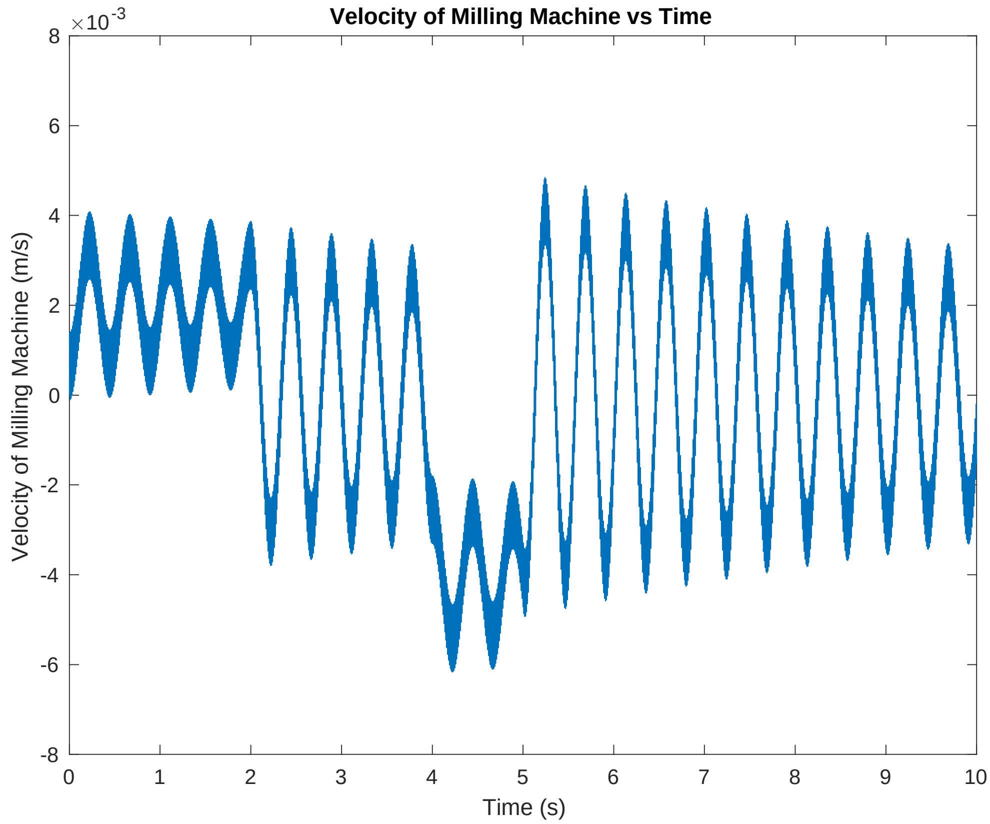


Figure 24-Velocity of Milling Machine vs Time, Given Input Parameters, All Initial Conditions 0, 500,000 Increments Considered

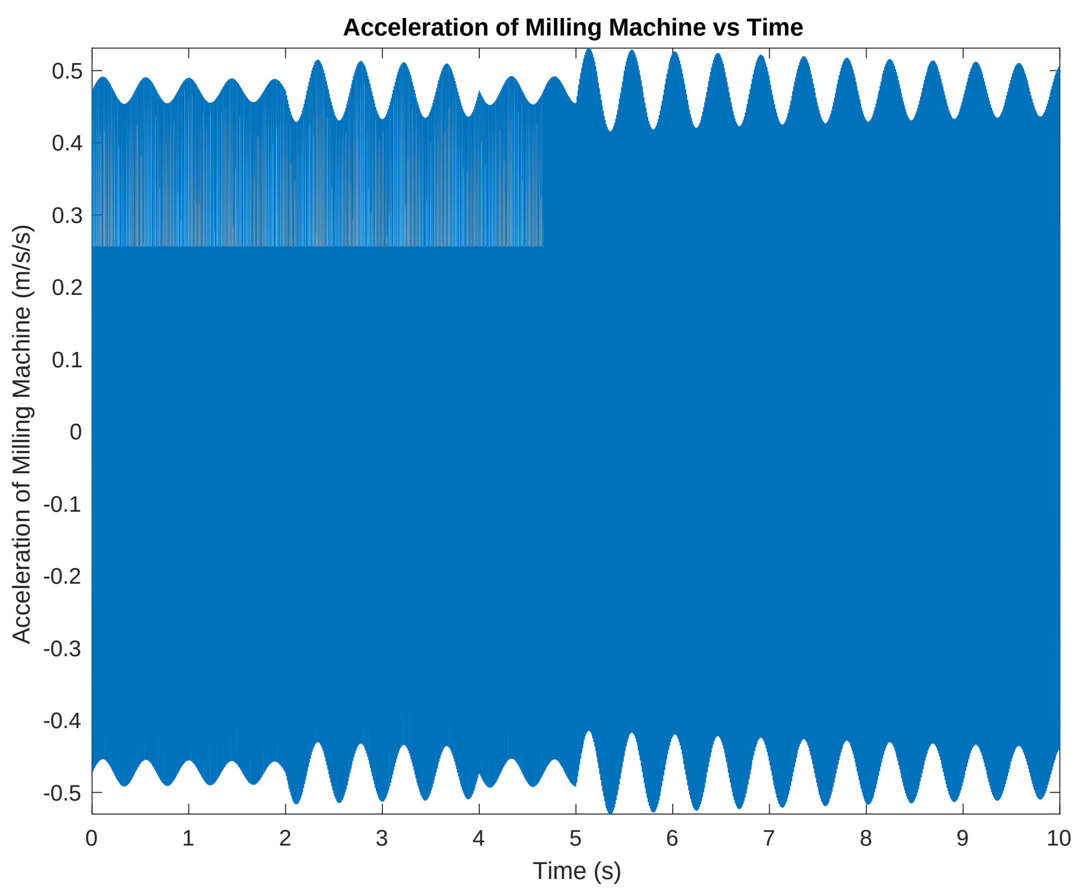


Figure 25-Acceleration of Milling Machine vs Time, Given Input Parameters, All Initial Conditions 0, 500,000 Increments Considered

This resulted in the following values for the magnitude of variation in position, velocity, and acceleration, respectively:

1.) 0.0045125522457674580034669986616791 (m)
2.) 0.011015434925615821132316440866816 (m/s)
3.) 1.061128050376433318291446994408 (m/s/s)

Comparing these results with the results gathered when 100,000 increments were considered, the following values represent the percent error/deviation between the results for the magnitude of variation in position, velocity, and acceleration when 100,000 and 500,000 increments were considered

1.) Percent error in position=9.8758*10^{-6}%

2.) Percent error in velocity=.0025%

3.) Percent error in acceleration=.0647%

This shows that the model converges when enough increments are considered. It also demonstrates that 100,000 increments is adequate to accurately capture the behavior of the system.

Conclusion:

The objective of this project was completed by deriving the equation of motion for the milling machine, symbolically solving for the position of the milling machine as a function of frequency in the Laplace domain, solving the symbolic expression for the position of the milling machine as a function of frequency by taking the Laplace transform computationally, then computationally taking the inverse Laplace transform to determine the position of the milling machine as a function of time. The parameters influencing the model were then modified and the result was compared to what would be expected. The model behaved exactly as would be expected for each case considered. Next, the resolution of the model was increased until it converged, once again proving the model behaves as it should. The final results shown in Figures 22-24 as well as the magnitude of variation in position, velocity, and acceleration were determined over a time period of 10 seconds, accomplishing the objective of the assignment. Additionally, a very important idea for vibration analysis was considered by analyzing the milling machines response when the frequency of the harmonic disturbance of the foundation was at the resonant frequency of the milling machine. This was considered for two different values of damping coefficient as well, demonstrating how critical it is to correctly size dampers to prohibit oscillation beyond a desired magnitude.

Overall, this project was a great blend of knowledge learned from system dynamics and experience in solving differential equations computationally.

References:

N/A for this project. Professor Kang ensured I was fully prepared to analyze this type of problem by covering all necessary content in ME-331: System Dynamics.

MATLAB Script:

```
%Mason Averill
%ME-544 Modeling and Simulation of ME Sytems
%Fall 2020
%09/04/2020


m=5000;%mass of milling machine in kg
c=1000;%damping coefficient of milling machine in N-s/m
k=10^6;%spring constant of milling machine in N/m
a=0; %initial position in x
b=0; %initial velocity in x
d=0; %initial position in y

syms s t

%enter y(t)

y=2/1000*sin(200*pi()*t);

%Resonant frequency y input
%y=2/1000*sin(sqrt(k/m)*t);

%F in Direction of x

F=2000*t*heaviside(t)-2000*t*heaviside(t-2)+4000*heaviside(t-2)-
4000*heaviside(t-4)+(-4000*t+20000)*heaviside(t-4)-(-
4000*t+20000)*heaviside(t-5);

%F opposed to x direction

%F=-(2000*t*heaviside(t)-2000*t*heaviside(t-2)+4000*heaviside(t-2)-
4000*heaviside(t-4)+(-4000*t+20000)*heaviside(t-4)-(-
4000*t+20000)*heaviside(t-5));



%Find laplace transform of F

laplace_F=laplace(F,t,s);

%Find Laplace transform of Y with Initial conditions

laplace_y=laplace(y,t,s);

laplace_y_with_ics=c*(s*laplace_y-d)+k*laplace_y;

%Represent RHS of equation of motion expression
rhs_with_ics=laplace_y_with_ics+laplace_F+m*s*a+m*b+c*a;

%Solve equation of motion expression for X(s)
x(s)=rhs_with_ics/(m*s^2+c*s+k);
```

```
%Take the inverse Laplace transform of X(s)
x(t)=ilaplace(x(s));

%Set the number of increments to be considered for the model,
%representative of the resolution of the model, with a higher value
%indicating a higher resoluion
number_of_increments=10000;

%Set the max time of interest for the equation of motion
max_time=10;

%create linearly spaced row vector to be utilized to determine x(t) for
%each element
t_sub=linspace(0,max_time,number_of_increments);

%Probably a better way of converting x(t) to a numeric array but this works
syms x1(t)
x1(t)=subs(x,{t},{t_sub});

out_sym=formula(x1);

numeric_x_t=double(out_sym);

%%now have x(t) as a numeric array

%built in matlab function to find dx
dx=diff(numeric_x_t);

%ditto but for dt
dt=diff(t_sub);

%Find velocity
v_t=dx./dt;


%Positon vs time plot
figure(1)

plot(t_sub,numeric_x_t)
title('Position of Milling Machine vs Time')
xlabel('Time (s)')
ylabel('Position of Milling Machine (m)')
axis([0 max_time -.005 .005])


%diff leaves array less one element(takes the difference between elements),
so adjust time array
t_sub(number_of_increments)=[];

%Velocity vs Time plot
figure(2)
plot(t_sub,v_t)
```

```
title('Velocity of Milling Machine vs Time')
xlabel('Time (s)')
ylabel('Velocity of Milling Machine (m/s)')
axis([0 max_time -.008 .008])

%diff leaves array less one element(takes the difference between elements, so
adjust time array
t_sub(number_of_increments-1)=[];

%diff leaves array less one element(takes the difference between elements, so
adjust dt array
dt(number_of_increments-1)=[];

dv=diff(v_t);

a_t=dv./dt;

%Acceleration vs Time plot
figure(3)
plot(t_sub,a_t)
title('Acceleration of Milling Machine vs Time')
xlabel('Time (s)')
ylabel('Acceleration of Milling Machine (m/s/s)')
axis([0 max_time -.08 .08])


%Find magnitude of variation in position,velocity, and acceleration
max_position_find=max(numeric_x_t);

min_position_find=min(numeric_x_t);

magnitude_of_variation_in_position=vpa(max_position_find-min_position_find)


max_velocity_find=max(v_t);

min_velocity_find=min(v_t);

magnitude_of_variation_in_velocity=vpa(max_velocity_find-min_velocity_find)

max_acceleration_find=max(a_t);

min_acceleration_find=min(a_t);

magnitude_of_variation_in_acceleration=vpa(max_acceleration_find-
min_acceleration_find)

disp('In m, m/s, and m/s/s, respectively')


% Resolution of model example

%t_for_y=linspace(0,10,100000);
```

```
%y_for_plot=2/1000*sin(200*pi()*t_for_y);

%figure(4)
%plot(t_for_y,y_for_plot)
%xlabel('Time (s)')
%ylabel('Y(t)')
%title('Y(t) vs Time')
%axis([0 10 -.003 .003])
```