To: D. W. Mueller, Jr., Ph.D., P.E.

From: Mason Averill

Subject: Main Project—Model Rocket Maximum Height Predictor

Date: 08 December 2020

Abstract:

The purpose of this project was to build a model to determine the expected maximum height a model rocket would reach given geometry of the rocket and specifications of the engine utilized for thrust. The model was to account for both change in mass due to use of propellant as well as variable drag force as a function of velocity. The desired resolution of model was to be accurate within 10% with many cases considered. The procedure to develop and ensure correctness of the model was to consist of the following distinct steps: mathematical modeling, verification, validation, and execution of the model/simulation. The model was to accept inputs pertaining to the geometry of the rocket, including diameter and mass, as well as engine specific-data. The mathematical modeling of the simulation exploited the computational solution method to be implemented by considering the acceleration to be constant during each small time increment considered. The validity of this assumption was to be investigated during the validation stage of analysis. Once a verified and validated model had been created, over 30 different model rockets, each consisting of a unique combination of geometry and mass, were to be simulated. The results from these simulations were to be compared to the manufacturer's anticipated maximum height achievable. Upon verification, validation, and execution of the simulation for over 30 unique model rockets, the percent error was found to be less than 7% for both the 'constant mass, constant drag' case considered as well as the 'variable mass, variable drag' case considered. Neglecting major outliers, the percent error dropped to less than 5.3% for both considered cases. With these results, including or excluding outliers, the objective of the project had been achieved.

Purpose:

The purpose of this memo is to communicate the methodology and results of the Main Project—*Model Rocket Maximum Height Predictor* for ME-544: Modeling and Simulation of Mechanical Engineering Systems, completed December 8th 2020.

Purpose and Scope of Assignment:

The purpose of this project was to build a model to determine the expected maximum height a model rocket would reach given geometry of the rocket and specifications of the engine utilized for thrust. Specifically, this project focused on ESTES Model Rockets[3] powered by ESTES C6[2] engines. Both variable drag force and change in mass of the rocket due to use of propellant were to be considered. The verified and validated model was to be executed and results from the model were to be compared to manufacturer claims about the anticipated maximum height a specific model rocket would reach. An average percent error of less than 10% was desired.

Mathematical Modeling of Problem:

This problem was modeled mathematically by utilizing Newton's Second Law and treating the acceleration as constant over a very small time step. This allowed for the implementation of Equations 1-3 to find the acceleration, velocity, and position, respectively, at each time step.

$$F_{Net} = m * a_i => a_i = \frac{F_{Net}}{m} \qquad \text{Equation 1}$$

With:

- $F_{Net} = net\ force\ (N)$
- $m = mass\ (kg)$
- $a_i = acceleration\ at\ current\ time\ step\ (m/s^2)$

$$V_j = V_i + a_i * \Delta t \qquad \text{Equation 2}$$

With:

- $V_j = velocity\ at\ beginning\ of\ next\ time\ step\ \ (m/s)$

- $V_i = velocity\ at\ beginning\ of\ current\ time\ step\ (m/s)$
- $a_i = acceleration\ at\ beginning\ of\ current\ time\ step\ (m/s^2)$
- $\Delta t = time\ step\ (s)$

$$S_j = S_i + V_i * \Delta t + \frac{1}{2} * a_i * \Delta t^2 \qquad \text{Equation 3}$$

With:

- $S_j = position\ at\ beginning\ of\ next\ time\ step\ \ (m)$
- $S_i = position\ at\ beginning\ of\ current\ time\ step\ (m)$
- $V_i = velocity\ at\ beginning\ of\ current\ time\ step\ (m/s)$
- $a_i = acceleration\ at\ beginning\ of\ current\ time\ step\ (m/s^2)$
- $\Delta t = time\ step\ (s)$

In order to find the net force acting on the rocket, a free body diagram was drawn and forces acting on the rocket were determined. These forces included:

- Force of gravity
- Drag force
- Thrust from the engine

The force of gravity was taken to be constant, as the maximum height predicted by the manufacturer for all of the rockets to be considered was less than 500 meters.

In order to determine the drag force, a drag coefficient first had to be found. Figure 1 (P. J. Pritchard and J. W. Mitchell, 2015) displays the drag coefficient vs Reynolds number for a smooth circular cylinder, which was used to find the drag coefficient for many considered Reynolds numbers. With this data, a linear interpolation could be utilized to determine the drag coefficient for any Reynolds number considered in Figure 1.

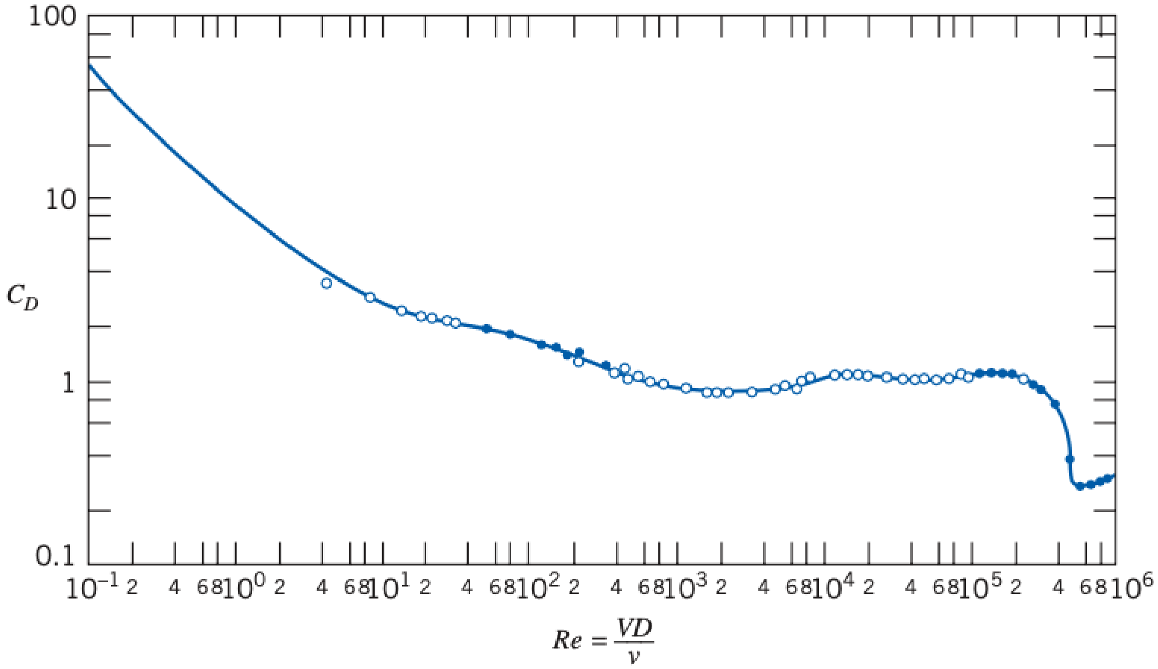

Figure 1: Drag Coefficient vs Reynolds Number for a Smooth Circular Cylinder

Once a drag coefficient is determined, Equation 4 can be utilized to determine the drag force.

$$C_D = \frac{F_D}{\frac{1}{2}\rho V^2 A} \Rightarrow F_D = \frac{1}{2}\rho V^2 A C_D \qquad \text{Equation 4}$$

With:

- $F_D = drag\ force\ (N)$
- $C_D = drag\ coefficient$
- $\rho = density\ of\ fluid\ (air, kg/m^3)$
- $V = velocity\ (m/s)$
- $A = maximum\ cross - sectional\ area\ (m^2)$

The only force yet to be quantified was the thrust from the engine. Thrust vs Time data for an Estes C6 Engine was obtained from *ThrustCurve* (Domansky, N., 2014). A linear interpolation was then used to determine the thrust of the engine for any time desired, shown by Figure 2.

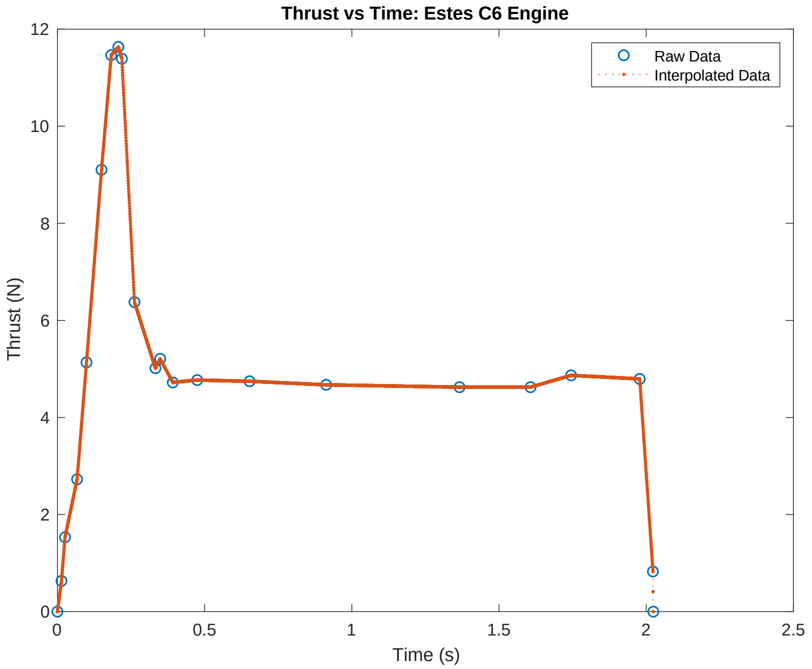

Figure 2: Thrust vs Time for Estes C6 Engine

With all forces quantified and their relation to the motion of the rocket established, the mathematical modeling of the problem had been completed.

<u>Verification of Model:</u>

Next, the mathematical model was developed into computational logic and coded in MATLAB. First, a simple case consisting of constant mass and no drag force was considered for an Estes 7220 Crossfire ISX model rocket, shown by Figure 3 with properties as given by Table 1.

Figure 3: Estes 7220 Crossfire ISX Model Rocket

Table 1-Estes Model Rocket Specifications[3]

Estes 7220 Crossfire ISX Specifications	
Diameter (input parameter)	25 mm
Mass of Rocket (input parameter)	37 grams
Expected Maximum Height	351 meters
Estes C6 Engine Specifications	
Total Mass (input parameter)	24 grams
Mass of Propellant (input parameter)	11 grams
Total Impulse	10 N*s

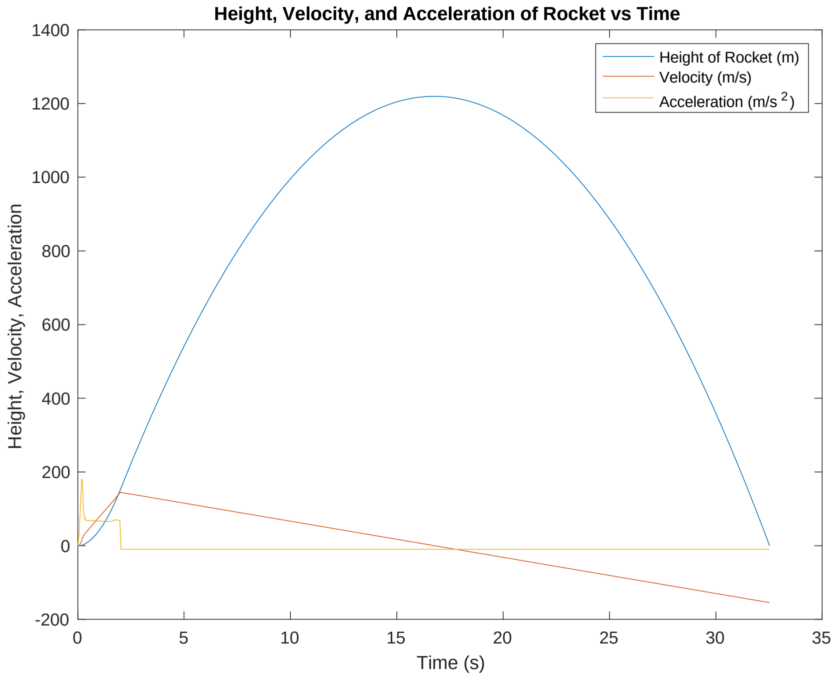

Figure 4: Estes 7220 Crossfire ISX Position, Velocity, and Acceleration vs Time--Constant Mass, No Drag Case

By viewing Figure 4, it is evident that the model produced an expected output. The acceleration vs time plot takes the form of the thrust vs time plot during the thrusting period of the rocket (time 0 to ~2 seconds), then shows a constant negative acceleration due to gravity for the remainder of the flight time. The maximum height is also achieved when the velocity goes to 0, which is to be expected. The velocity vs time and position (or height) vs time plots also appear as would be anticipated. However, without drag considered, it is evident that the model does not align well with the manufacturer's claim about the maximum anticipated height, as it predicts a maximum height of over 1200 meters when only 351 meters is expected.

Next, a constant mass and constant drag case was considered for the same rocket. The constant drag coefficient was taken to be 0.75[1]. Figures 5-6 show the output of the model under these conditions.

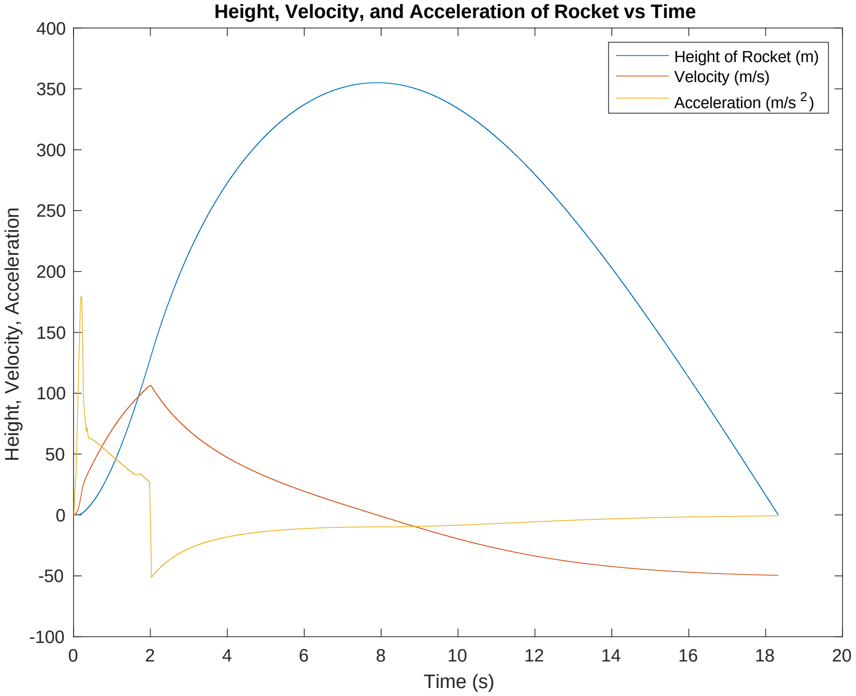

Figure 5: Estes 7220 Crossfire ISX Position, Velocity, and Acceleration vs Time--Constant Mass, Constant Drag Case

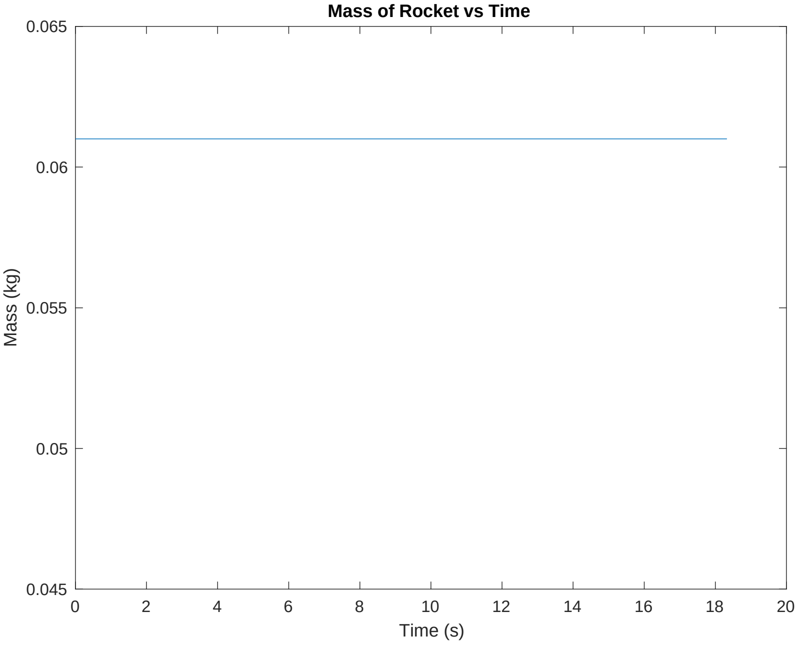

Figure 6: Estes 7220 Crossfire ISX Mass vs Time—Constant Mass Case

By reviewing Figure 5, again it is evident that the model produces an expected output. The acceleration vs time plot takes the form of the thrust vs time plot during the thrusting period of the rocket, but begins to quickly negatively accelerate when the thrusting ends—this is due to a large drag force at this point in time. This is exactly what would be anticipated, as the drag force is directly proportional to the square of the velocity of the rocket (shown by Equation 4), which would be anticipated to reach a maximum value after the thrusting period of the rocket ends. Figure 5 further confirms this expectation by showing that the maximum velocity is obtained at approximately 2 seconds, the end of the thrusting period of the rocket. In addition, the maximum height predicted by the model is 355.1 meters. This compares very well with the manufacturer’s claim of 351 meters.

It is also important to note how drastic of an impact drag force has on the maximum predicted height. Comparing Figure 5 to Figure 4 demonstrates just how critical considering drag force is for problems of this type.

Now that confidence in the model’s accuracy had been established, the complexity of the analysis was increased by also considering variable mass of the rocket as the engine uses its propellant. In order to determine the remaining amount of propellant in the engine, numerical integration methods were utilized. The total area under the Impulse vs Time plot shown in Figure 2 was found to be 10.02 N*s, which matches very well with the manufacturer’s claim of 10 N*s, further increasing both the validity of the model and the data used to construct the Thrust vs Time plot. With the total mass of the propellant known, the total mass of the propellant used at any time during the thrusting period could be found by using Equation 5.

$$\text{Curent Mass of Propellent Used} = \frac{Current\ Area}{Total\ Area} * Total\ Mass\ of\ Propellant \qquad \text{Equation 5}$$

With:

- $Current\ Area = current\ area\ under\ the\ Thrust\ vs\ Time\ plot$
- $Total\ Area = total\ area\ under\ the\ Thrust\ vs\ Time\ plot$

Referring to Figure 1, some choices regarding the drag coefficient vs Reynolds number had to be made. After much experimentation, it was found that the drag coefficient of a typical model rocket is approximately 75% of that for a smooth cylinder. This is accomplished by the addition of end caps and other aerodynamic considerations during the design stage of the model rocket. In addition, model rockets frequently reach a very high velocity, exceeding the Reynolds number bounds shown in Figure 1. While it initially may be expected that using the drag coefficient coinciding with the largest shown Reynolds number in Figure 1 would produce accurate results, this is not the case. This is because the maximum velocity obtained by model rockets can be in excess of one half of the speed of sound, at which point they could be approaching or surpassing the drag-divergence Mach number. Due to this, the model used a drag coefficient of 0.6 when the Reynold's number surpassed the upper bound shown in Figure 1. This value was obtained after much experimentation with various model rocket geometries were considered.

Finally, Figures 7-8 show the results of running the model with both variable mass and variable drag considered.

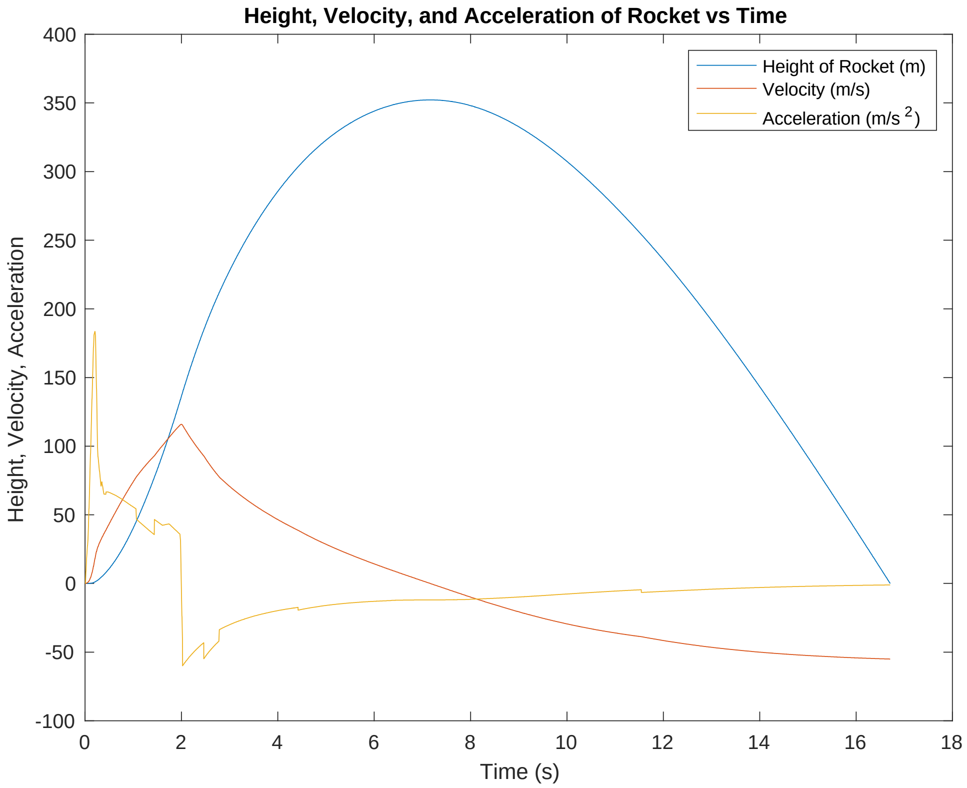

Figure 7: Estes 7220 Crossfire ISX Position, Velocity, and Acceleration vs Time--Variable Mass, Variable Drag Case

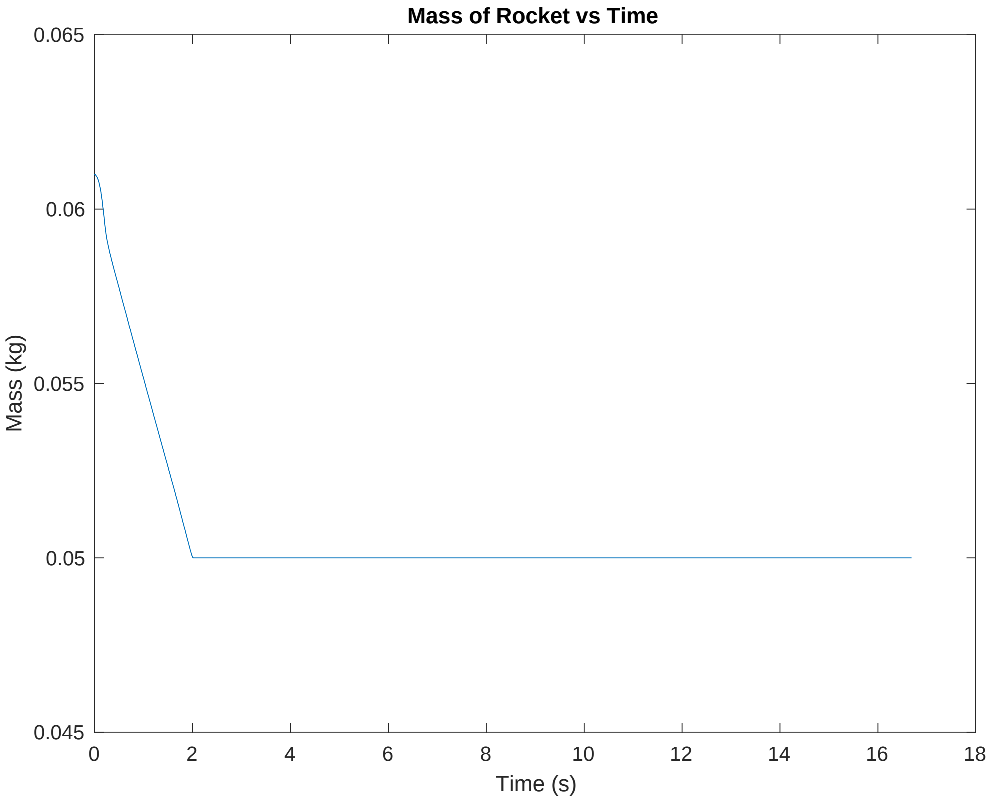

Figure 8: Estes 7220 Crossfire ISX Mass vs Time—Variable Mass Case

Figure 7 indicates that the model predicts a maximum height of 352.0 meters to be obtained. This matches incredibly well with the manufacturer's prediction of 351.0 meters, a percent error of just 0.3%. Figure 8 shows the mass of the rocket vs time. As would be anticipated, the mass is reduced during the thrusting period, then remains constant during the remaining flight time, further verifying the model.

Validation of Model:

The only potential pitfall determined for this model is selecting too coarse of a time step—at which point both the constant acceleration behavior during each time step is no longer valid and the functions utilized to generate the linearly interpolated points fall apart. To determine the point at which the step size becomes 'too coarse', all inputs into the model were kept constant besides the step size and the results were plotted. This is shown by Figure 9.

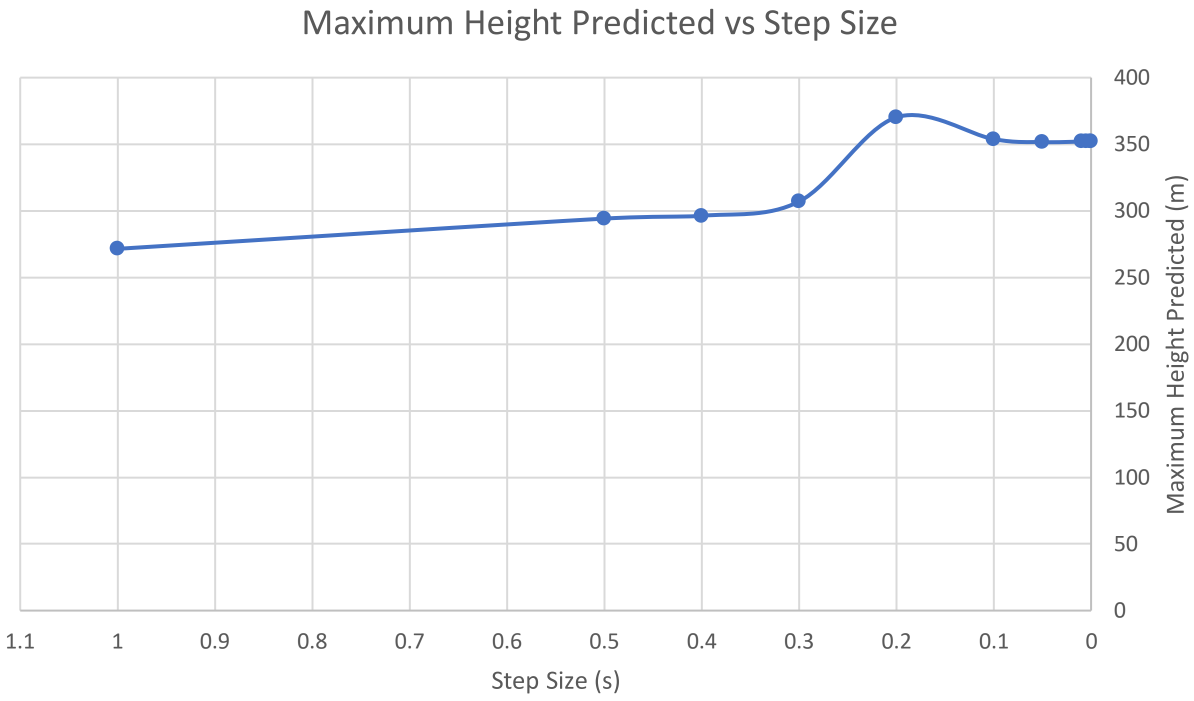

Figure 9: Estes 7220 Crossfire ISX Maximum Height Predicted vs Step Size--Variable Mass, Variable Drag Case

Figure 9 indicates that a step size larger than 0.1 seconds should not be utilized, as this seems to be the point at which the results begin to converge. As long as the step size is chosen to be smaller than 0.1 seconds, the physics the model is built on is valid.

Execution of Simulation to Determine The Maximum Height Predicted for Numerous Model Rockets:

A total of 33 different Estes Model Rockets were considered, each consisting of a different combination of geometry and mass. All considered model rockets utilized the same C6 engine type. Both the ‘constant mass and constant drag case’ and the ‘variable mass and variable drag case’ were considered. All simulations were run with a time step of 0.01 seconds, ensuring that the model was valid. All other parameters in the model besides the rocket geometry and mass were kept constant between each simulation. Figures 10-16 and Table 2 depicts these results.

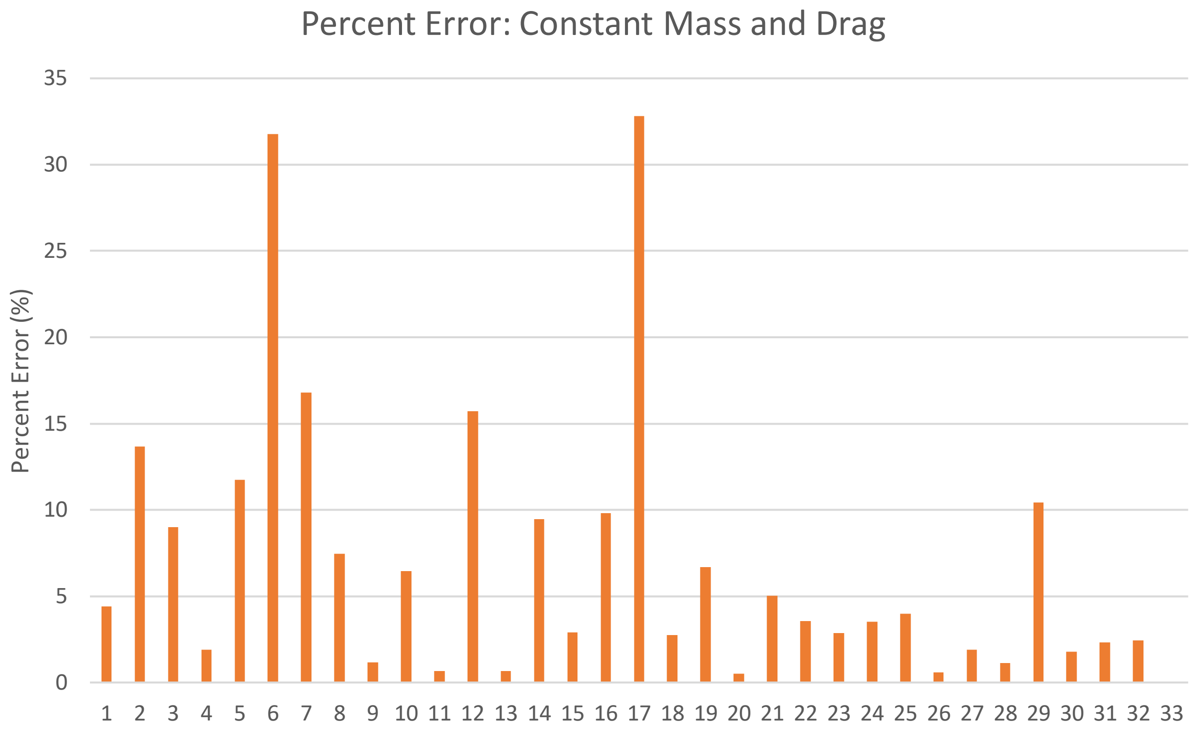

Figure 10: Percent Error--Constant Mass, Constant Drag Case

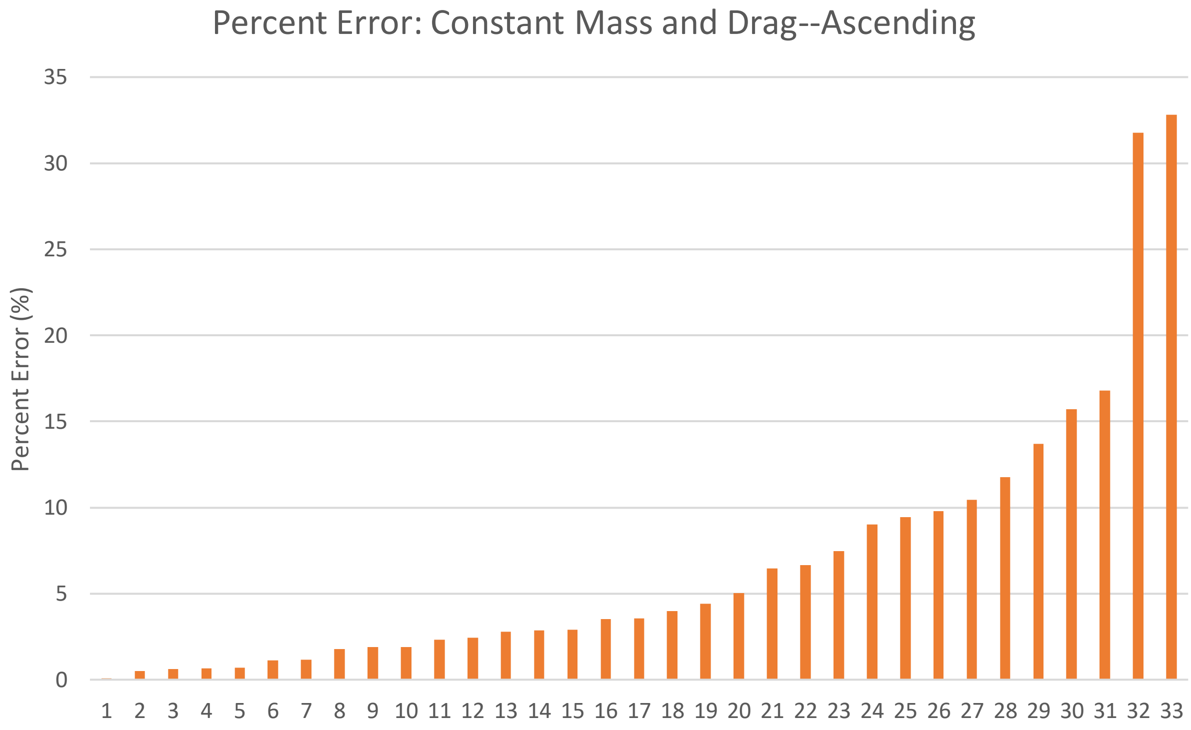

Figure 11: Percent Error--Constant Mass, Constant Drag Case, Ascending Order

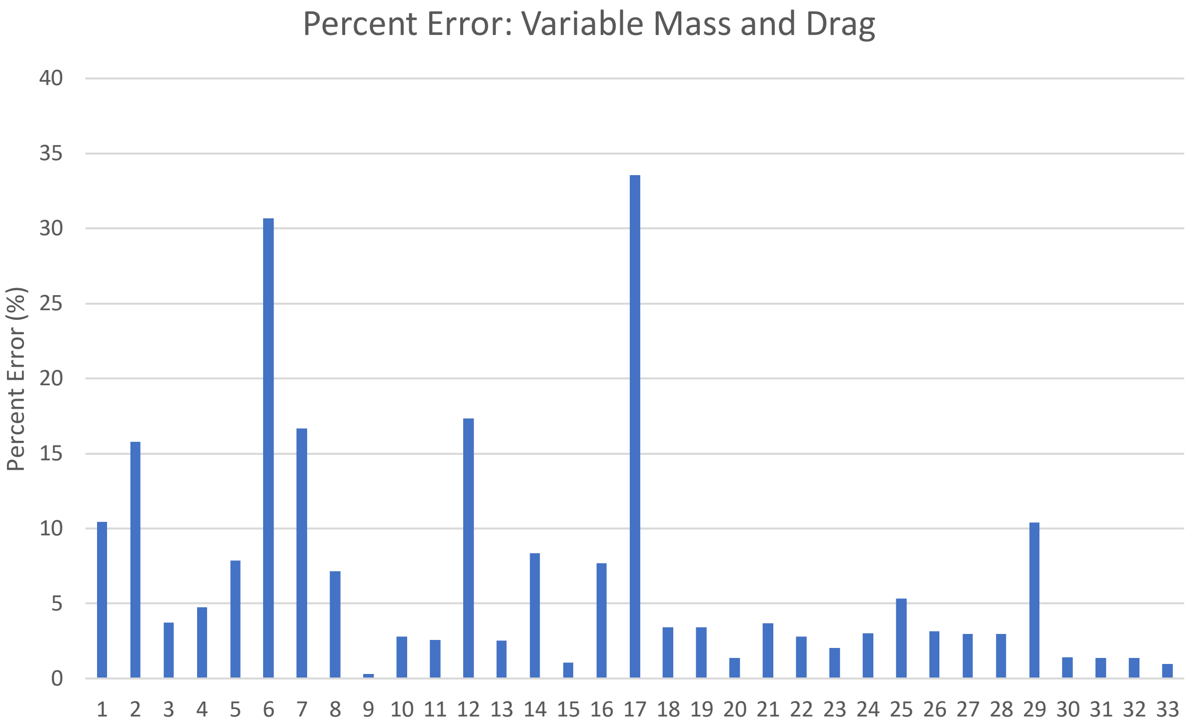

Figure 12: Percent Error--Variable Mass, Variable Drag Case

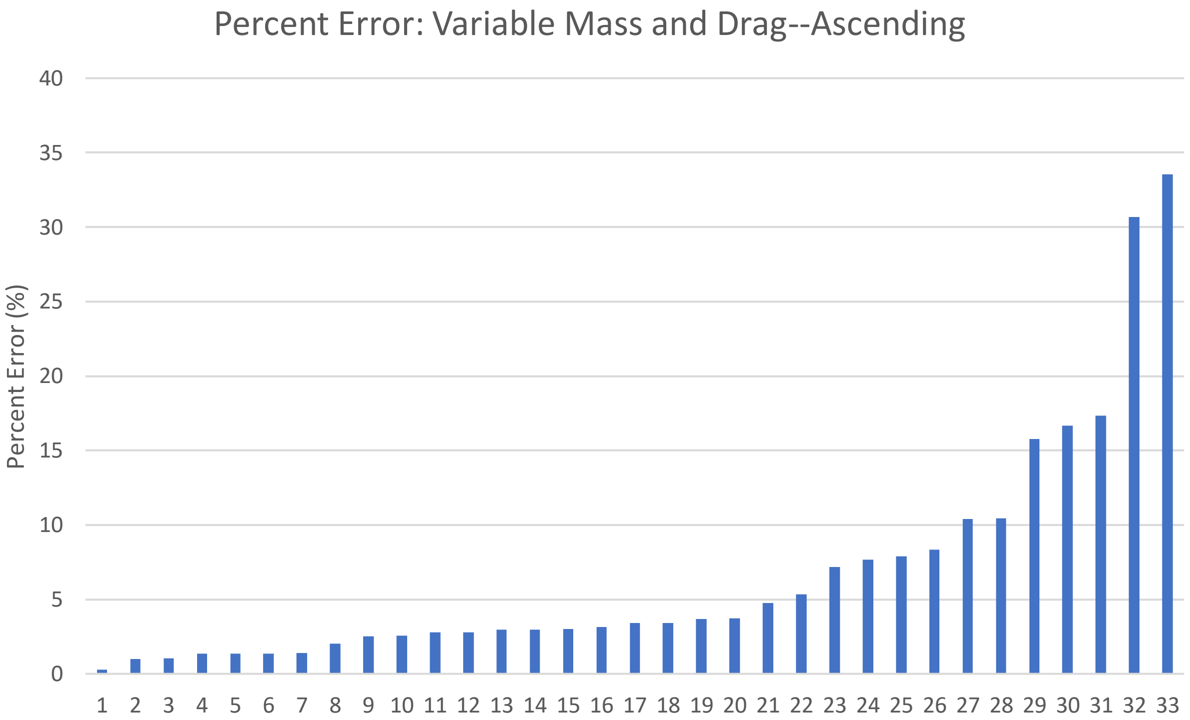

Figure 13: Percent Error--Variable Mass, Variable Drag Case, Ascending Order

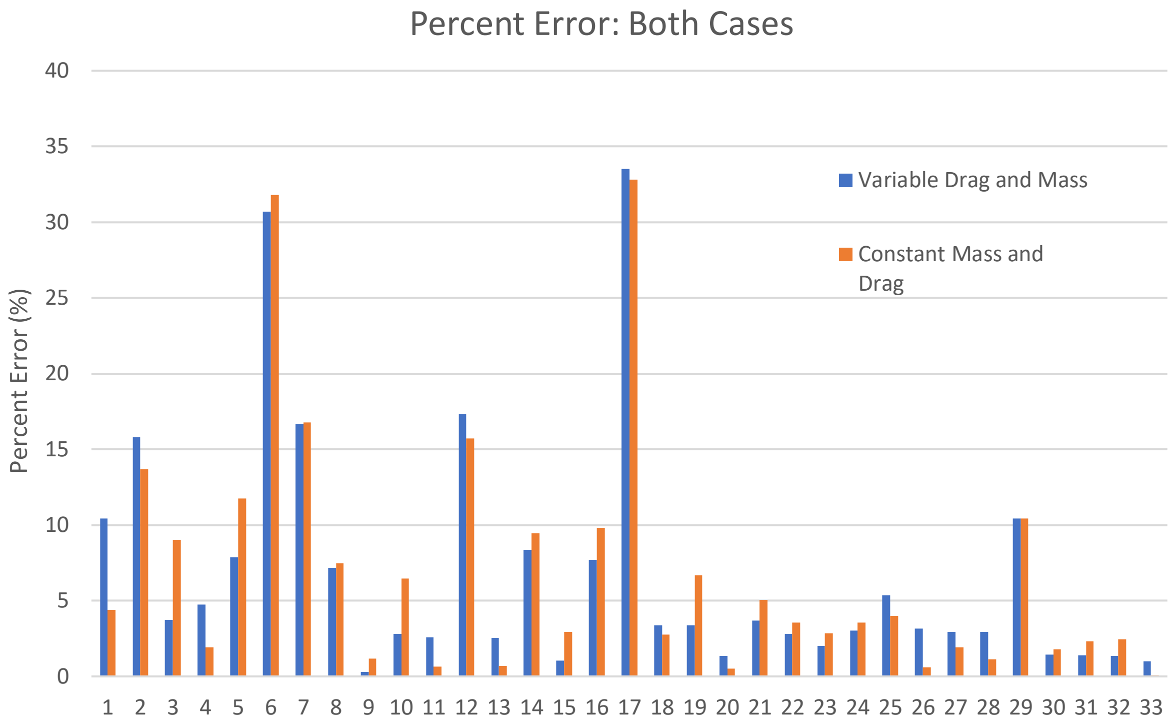

Figure 14: Percent Error--Both Cases

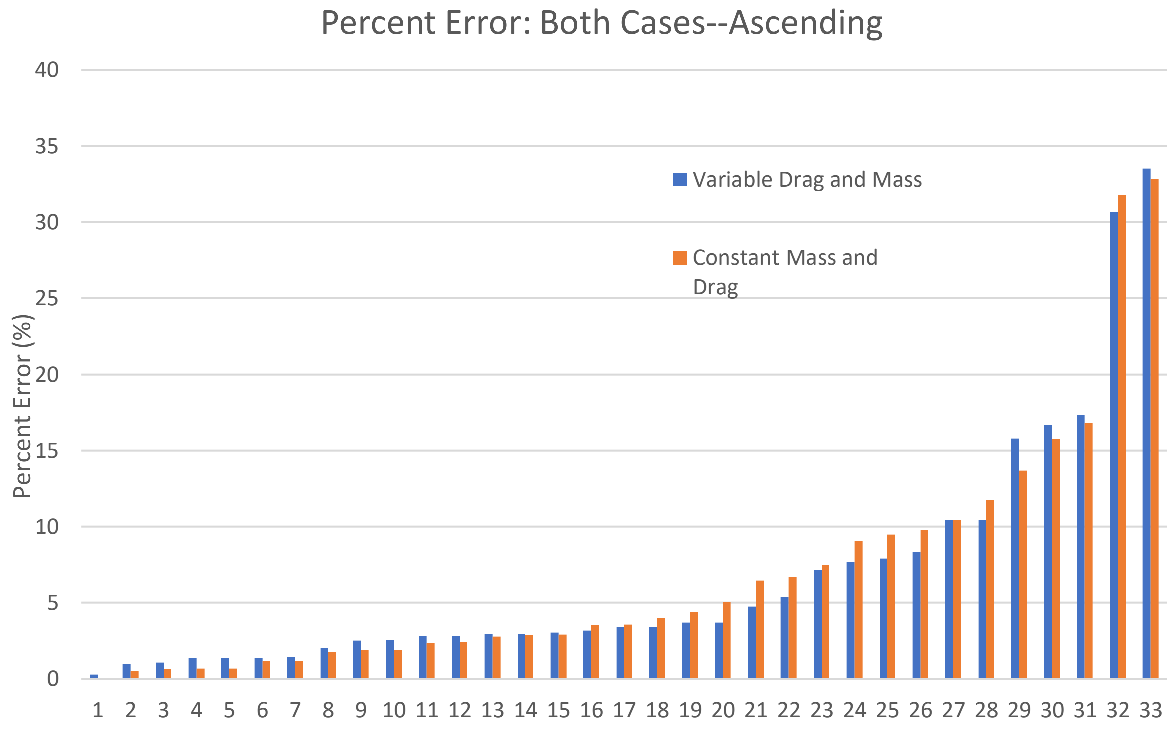

Figure 15: Percent Error--Both Cases, Ascending

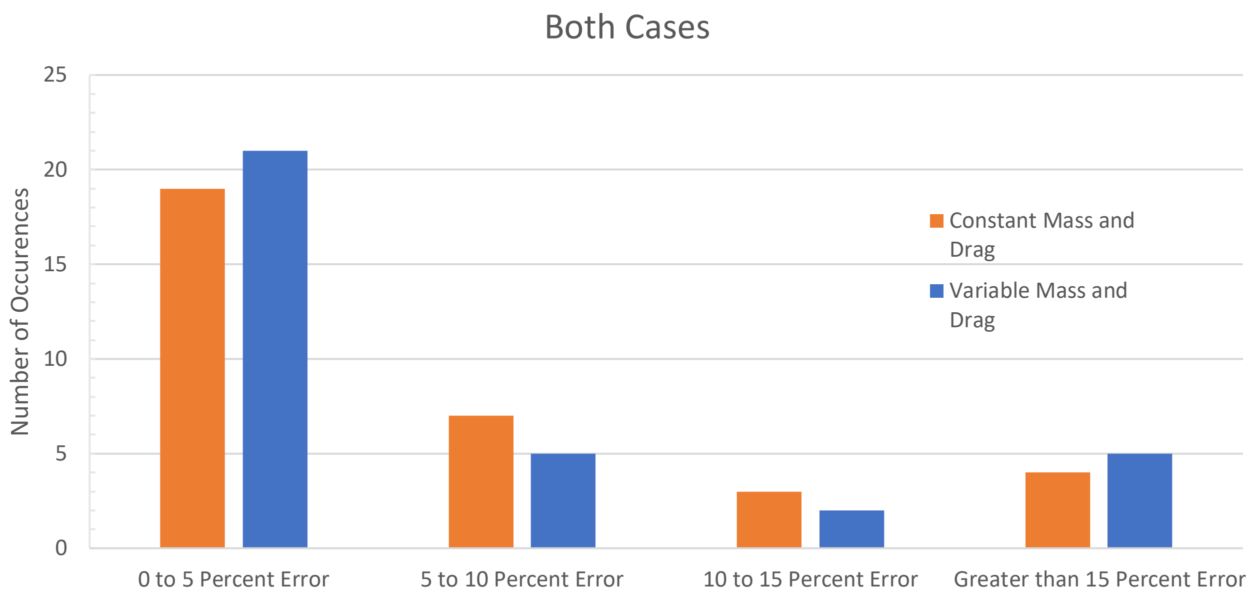

Figure 16: Percent Error--Both Cases, Histogram

Table 2-Simulation Results

Percent Error Statistics		
Data Parameter	Constant Mass, Constant Drag Case	Variable Mass, Variable Drag Case
Average Percent Error (%)	6.85	6.75
Average Percent Error, Less Outliers** (%)	5.21	5.12
Standard Deviation (%)	8.00	7.96
Standard Deviation, Less Outliers** (%)	4.72	4.67

**Outliers were taken to be trials in which the percent error exceeded 30%

By reviewing Figures 10-16 and Table 2, it is evident that both the constant mass and constant drag case as well as the variable mass and variable drag case produce similar, precise and accurate results. The average percent error for both considered cases is less than 7%. Neglecting outliers, the average percent error is less than 5.3%. The standard deviations in the percent errors is low and quite comparable for both cases as well. The deviation in the accuracy of the model is suspected to be due to variations in the drag coefficient specific to each model rocket. A generalized model was created to be representative of model rockets as a whole, it is not specific to any one particular model rocket, and thus is not capable of having a *high* degree of accuracy (e.g. less than 1% error on average).

Conclusion:

The objective of this project was accomplished by mathematically deriving and assembling equations governing the motion of a model rocket. Verifying that the mathematics were correctly translated into a machine code and were capable of considering both variable mass and variable drag. Validating that the physics the model relies on is accurate and representative of the situation. And finally executing the simulation and comparing its output to the manufacturer's claim regarding the maximum height a model rocket will achieve. Numerous model rockets were considered to ensure that the model is both accurate and precise. Of the two different cases considered, 'constant mass, constant drag' and 'variable mass, variable drag', both resulted in an average percent error of less than 7%, meeting the goal of a percent error of less than 10%. This is especially the case when outliers are not considered, at which point the average percent error drops to below 5.3% for both cases considered.

References:

1. Benson, T. (2014, June 12). Shape Effects on Drag. Retrieved December 09, 2020, from https://www.grc.nasa.gov/www/k-12/rocket/shaped.html
2. Domansky, N. (2014, December 3). ThrustCurve Hobby Rocket Motor Data. Retrieved December 09, 2020, from https://www.thrustcurve.org/motors/Estes/C6/
3. *ESTES 2016 Catalog* [PDF]. (2016). Penrose, CO: 2016 Estes-Cox Corporation.
4. P. J. Pritchard and J. W. Mitchell. *Fox and McDonald's: Introduction to Fluid Mechanics.* 9th ed. Wiley, New York (2015).

MATLAB Code:

```
%Mason Averill
%ME-544 Fall 2020, 10/31
%Main Project--Estes Rockets Expected Height Achieved

%Rocket Parameters
Diameter=25/1000;%Diameter in meters
Mass_rocket=37/1000;%mass of rocket in kg
Mass_engine=24/1000;%mass of rocket engine(including propellant)
Mass_propellent=11/1000;%mass of propellant in rocket engine



Mass=Mass_rocket+Mass_engine;%mass in kg
Weight=Mass*9.81;%Weight in N
Cross_sectional_area=pi*Diameter^2/4;%effective area

%Air Properties
density_air=1.225;%kg/m^3
dynamic_viscosity=1.81*10^-5;


%Desired Resolution
time_steps=5*10^-2;

%Engine type:Estes c6 Raw Data

time=[0 0.014   0.026   0.067   0.099   0.15    0.183   0.207   0.219   0.262
0.333   0.349   0.392   0.475   0.653   0.913   1.366   1.607   1.745   1.978
2.023   2.024];

thrust=[0 0.633 1.533   2.726   5.136   9.103   11.465  11.635  11.391  6.377
5.014   5.209   4.722   4.771   4.746   4.673   4.625   4.625   4.868   4.795
0.828   0];

time_range=0:time_steps:max(time);

linearly_interpolated_points=interp1(time,thrust,time_range);


figure(1)
plot(time,thrust,'o',time_range,linearly_interpolated_points, ':.')
xlabel('Time (s)')
ylabel('Thrust (N)')
legend('Raw Data','Interpolated Data')
title('Thrust vs Time: Estes C6 Engine')
grid on

impulse=trapz(time_range,linearly_interpolated_points);
impulse_of_t=cumtrapz(time_range,linearly_interpolated_points);

%Cd determination/curve fitting
Re_raw=[0.1 10 100 1000 6*10^3 10^4 2*10^5 3*10^5 4*10^5 5*10^5 10^6];
```

```
Cd_raw=[38 2.4 1.8 0.98 .98 1.1 1.1 .98 .45 .25 .27];

%coefficients=polyfit(Re_raw,Cd_raw,degree);

Re_range=0.1:0.1:10^6;

linearly_interpolated_Cd=interp1(Re_raw,Cd_raw,Re_range);%or linearly
interpolated method



flight_time_store=[];
position_store=[];
velocity_store=[];
acceleration_store=[];
position=0;
velocity=0;
acceleration=0;
position_store(1,1)=position;
velocity_store(1,1)=velocity;
acceleration_store(1,1)=acceleration;
F_net_store=[];
F_net=0;
flight_time=0;
flight_time_store(1,1)=0;
i=1;
%c_d=0.75;
%c_d=0;
Current_mass=Mass;
Current_mass_store=[];
while(position>=0)
    %determine c_d
  Re=density_air*abs(velocity_store(1,i))*Diameter/dynamic_viscosity;
  Re=round(Re);
  [row,column]=find(Re==Re_range);
  if(row==1)%this Re exists in possible Re values considered
     c_d=0.75*linearly_interpolated_Cd(1,column);

  elseif(Re<0.1)
     c_d=Cd_raw(1,1);
      %c_d=0.60;
  else
      %c_d=Cd_raw(1,11);
     c_d=0.6;
   end

  %no liftoff until thrust exceeds weight of rocket
  if(flight_time<0.1 && linearly_interpolated_points(1,i)<Weight)
     F_net=0;
     Current_mass=Mass-(impulse_of_t(i)/impulse)*Mass_propellent;

  %thrust being generated and liftoff has occured
  elseif(flight_time<=max(time))

     if(velocity_store(1,i)>0)
```

```
        F_d=-
0.5*c_d*density_air*Cross_sectional_area*(velocity_store(1,i))^2;
     else
        F_d=0.5*c_d*density_air*Cross_sectional_area*(velocity_store(1,i))^2;
     end
      F_net=linearly_interpolated_points(1,i)-Weight+F_d;
     Current_mass=Mass-(impulse_of_t(i)/impulse)*Mass_propellent;
   %no thrust is being generated
  else
     if(velocity_store(1,i)>0)
        F_d=-
0.5*c_d*density_air*Cross_sectional_area*(velocity_store(1,i))^2;
    else
        F_d=0.5*c_d*density_air*Cross_sectional_area*(velocity_store(1,i))^2;
    end

    F_net=-Weight+F_d;
    Current_mass=Mass-Mass_propellent;
  end



acceleration=F_net/Current_mass;
velocity=velocity+acceleration*time_steps;
position=position+velocity*time_steps+0.5*acceleration*time_steps^2;
position_store(1,i+1)=position;
velocity_store(1,i+1)=velocity;
acceleration_store(1,i+1)=acceleration;
flight_time_store(1,i+1)=flight_time;
F_net_store(1,i)=F_net;
Current_mass_store(1,i)=Current_mass;

i=i+1;
flight_time=flight_time+time_steps
end

figure(2)
plot(flight_time_store,position_store)

hold on
plot(flight_time_store,velocity_store)

hold on
plot(flight_time_store,acceleration_store)

legend('Height of Rocket (m)','Velocity (m/s)','Acceleration (m/s^2)')

xlabel('Time (s)')
ylabel('Height, Velocity, Acceleration')
title('Height, Velocity, and Acceleration of Rocket vs Time')
grid on

figure(3)
plot(flight_time_store,position_store)
xlabel('Time (s)')
ylabel('Height (m)')
```

```
title('Height of Rocket vs Time')
ylim([0 400])
grid on


figure(4)
plot(flight_time_store,velocity_store)
xlabel('Time (s)')
ylabel('Velocity (m/s)')
title('Velocity of Rocket vs Time')
grid on


figure(5)
plot(flight_time_store,acceleration_store)
xlabel('Time (s)')
ylabel('Acceleration (m/s^2)')
title('Acceleration of Rocket vs Time')
grid on




max(position_store)
max(velocity_store)
flight_time_store(i)=[];

figure(6)
plot(flight_time_store,Current_mass_store);
xlabel('Time (s)')
ylabel('Mass (kg)')
title('Mass of Rocket vs Time')
ylim([0.045 .065])
grid on
```